

**DESIGN OF PLATE
AND SHELL STRUCTURES**

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By Maan H. Jawad, Ph.D., P.E.

President

Global Engineering & Technology



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Preface

The design of many structures such as pressure vessels, aircrafts, bridge decks, dome roofs, and missiles is based on the theories of plates and shells. The degree of simplification needed to adopt the theories to the design of various structures depends on the type of structure and the required accuracy of the results. Hence, a water storage tank can be satisfactorily designed using the membrane shell theory, which disregards all bending moments, whereas the design of a missile casing requires a more precise analysis in order to minimize weight and materials. Similarly, the design of a nozzle-to-cylinder junction in a nuclear reactor may require a sophisticated finite elements analysis to prevent fatigue failure while the same nozzle in an air accumulator in a gas station is designed by simple equations that satisfy equilibrium conditions.

A book that covers all aspects of plate and shell theory is impractical to write. Some books cover specific theories such as bending of plates, buckling, vibration, or shell theory while others cover specific topics such as concrete structures, roofs, water tanks, or pressure vessels. Some books use vector notations while others use standard derivation to arrive at the intended equations. Some books concentrate on theoretical derivations while others show extensive tabular data for particular applications. All of these books, put together, cover the wide field of the theory of plates and shells. Accordingly, professionals and organizations use specific books to fulfill their specific needs and requirements.

This book is written primarily for professional engineers interested in designing plate and shell structures. It covers basic aspects of the theories and gives examples for the design of components due to internal and external loads as well as other loads such as wind and dead loads. Various derivations are kept relatively simple and the resultant equations are simplified to a level where the engineer may apply them directly to design problems. More elaborate derivations and more general equations can be found in the literature for those interested in a more in-depth knowledge of the theories of plates and shells. The examples given throughout this book are intended to show the engineer the level

of analysis needed to achieve a safe design based on a given required degree of accuracy.

The theory of plates is discussed in the first seven chapters of the book. Chapters 1 and 2 cover rectangular plates with various boundary and loading conditions. Chapter 3 develops the theory of circular plates of uniform and variable thickness as well as plates on elastic foundation. Chapter 4 discusses the bending of plates with various shapes as well as the bending of orthotropic plates. Chapter 5 presents approximate analyses using energy, yield-line, and finite difference methods for evaluating plates of unique geometries. Chapter 6 addresses the buckling theory of plates and Chapter 7 gives a brief discussion of the vibration of plates.

Chapters 8 through 15 are dedicated to shell theory. Chapters 8 and 9 cover the membrane theory and its application to spherical, conical, and other shell configurations. Bending of cylindrical shells is discussed in chapter 10. Both long and short cylinders are analyzed for mechanical as well as thermal loads. Examples combining circular plates and cylindrical shell components are given to illustrate the design of some actual structures. Bending of shells with double curvature is discussed in Chapter 11 and numerical examples are given. Chapter 12 gives a brief introduction to shallow shell theory. Buckling of cylindrical shells is given in Chapter 13 while Chapter 14 discusses the buckling of spherical shells. A brief introduction to the vibration of shells is given in Chapter 15.

A discussion of plate and shell theories is incomplete without a brief mention of the finite element method given in Chapter 16. Many complicated structures have to be solved by the finite element method since a theoretical approach is impractical. However, a detailed presentation of the method is beyond the scope of this book. Thus, only a brief introduction is given in this chapter.

Most of the chapters in this book can be covered in a two-semester course in "Plates and Shells". Also, a special effort was made to make the chapters as independent from each other as possible so that a course in "Plate Structures" or "Shell Structures" can be taught in one semester by selecting the appropriate chapters.

In order to study and use the theory of plate and shells, the engineer is assumed to have a good working knowledge of differential calculus and matrix analysis. Some of the differential equations require solutions involving Fourier and Bessel functions. These two functions are presented in Appendices A and B at the end of the book for easy reference. Also, a summary of matrix operations is given in Appendix C and some conversion factors in English and SI units are given in Appendix D.

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St. Louis, MO
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Abbreviations for Organizations

AASHTO, American Association of State Highway and Transportation Officials

ACI, American Concrete Institute

AISC, American Institute of Steel Construction

AISI, American Iron and Steel Institute

API, American Petroleum Institute

ASME, American Society of Mechanical Engineers

AWWA, American Water Works Association

NASA, National Aeronautics and Space Administration

TEMA, Tubular Exchangers Manufacturers Association

*To
Jennifer and Mark
Who Taught Me A Lot*

Bending of Simply Supported Rectangular Plates

1-1 Introduction

Many structures such as boiler casings (Fig. 1-1), submarine bulkheads, ship and barge hulls, building slabs, (Fig. 1-2), aircraft components, and machine parts are designed in accordance with the general bending theory of plates. In this and subsequent chapters, the pertinent equations of the bending theory of plates are developed and examples are solved to demonstrate its applicability. These equations are applicable to thin plates subjected to small deflections. The majority of industrial applications encountered by the engineer fall under this category. Other theories dealing with thin plates with large deflections such as diaphragms; thick plates such as some tubesheets in heat exchangers; and composite and laminated plates in some aircraft components are beyond the scope of this book. Many of the references given at the end of the book delve into such theories and the interested reader is encouraged to read them.

The derivation of the pertinent equations for the bending of thin plates with small deflections is based on the following assumptions

1. The plate is assumed thin; i.e. the plate thickness is substantially less than the lateral dimensions. An approximate rule of thumb is the ratio of thickness to lateral dimension is of the order of about 0.1. This assumption is made in order to develop simplified equations applicable to a large variety of plates. As the ratio of thickness to lateral dimension increases, the stresses calculated by the thin plate theory become less conservative due to increased effect of the shearing forces, which are neglected in thin plate theory (Timoshenko and Woinowsky-Krieger 1959).
2. Applied loads are perpendicular to the middle surface of the plate. This assumption eliminates the need to consider in-plane membrane forces that are not considered in the classical theory of bending of thin plates. In-plane forces will be discussed briefly in Chapter 6 that deals with buckling of plates.

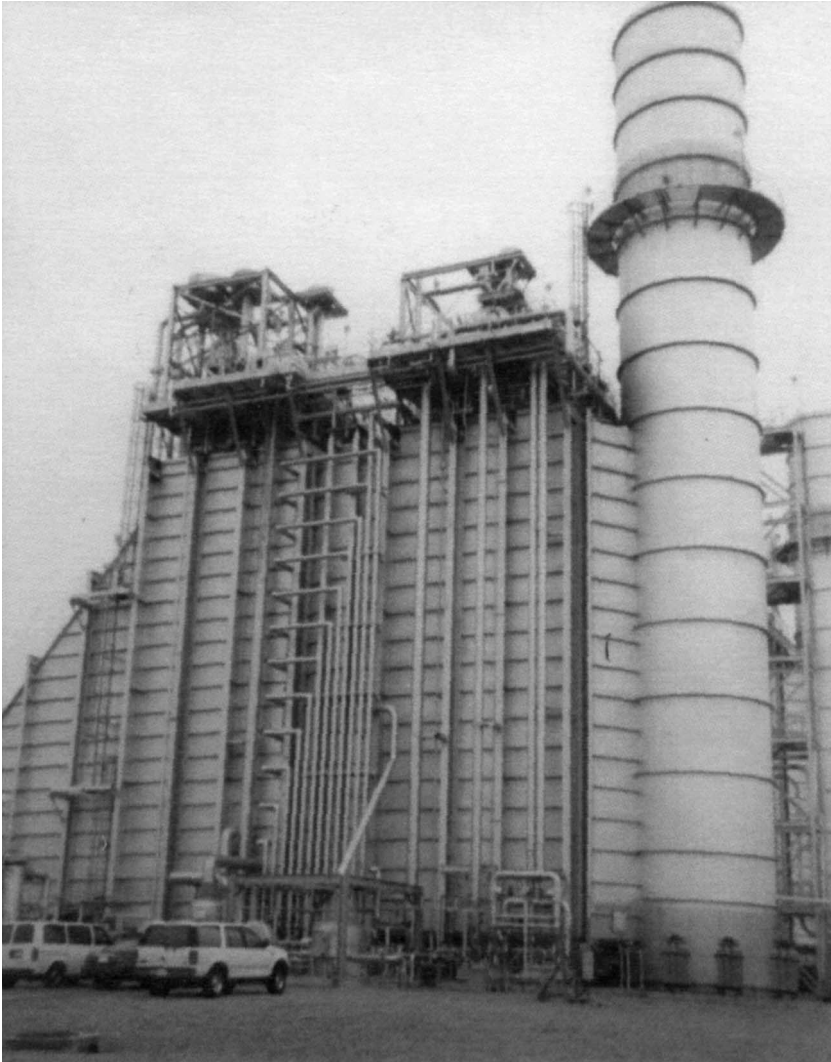


Figure 1-1. Boiler casing. (Courtesy of the Nooter Corporation, St. Louis, MO.)

3. The plate undergoes small deflections due to applied loads. Small deflections are defined as being less than the thickness of the plate. An approximate rule of thumb for defining small deflections is that the ratio of deflection to thickness is less than about 0.5. As this ratio increased above 0.5, the equations generally become overly conservative due to the introduction of



Figure 1-2. Reinforced concrete building. (Courtesy of the Portland Cement Association, Chicago, IL.)

membrane forces in the plate that tend to reduce the bending stresses predicted by the small deflection plate theory (Timoshenko and Woinowsky-Krieger 1959).

4. The undeflected middle surface of the plate coincides with the chosen x - y plane as shown in Fig. 1-3. Points on the middle surface undergoing small deflections due to transverse loads on the plate are assumed to move perpendicular to the x - y plane to form a new middle surface.
5. Cross-sectional planes perpendicular to the middle surface prior to deflection remain straight and perpendicular to the middle surface subsequent to its deflection. This implies that the middle surface does not undergo any extension and that the middle surface is also the neutral surface of the plate.
6. The plate material is isotropic and homogeneous. This assumption results in equations that are greatly simplified and easy to apply to a vast variety of plate components. A brief discussion of orthotropic plates is given in Chapter 4.

1-2 Strain-Deflection Equations

The relationship between strain and deflection of a thin plate is available from geometric considerations. We begin the derivation by letting the infinitesimal

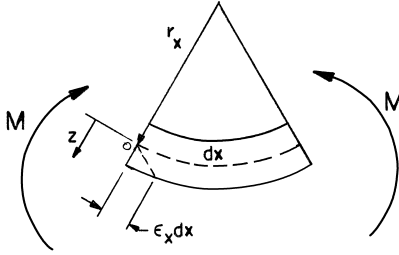
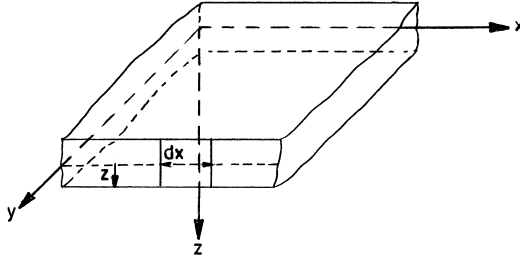


Figure 1-3. Bending of a rectangular plate.

section in Fig. 1-3 undergo some bending deformation. Downward deformation is defined as positive. The change in length at a distance z from the middle surface is obtained from the figure as

$$\frac{dx}{r_x} = \frac{dx + \epsilon_x dx}{r_x + z} \quad (1-1)$$

where,

dx = infinitesimal length in the x-direction

ϵ_x = strain in the x-direction

r_x = radius of curvature of the deformed middle surface in the x-direction

z = distance from the middle surface

Equation (1-1) may be simplified as

$$\epsilon_x = \frac{z}{r_x} \quad (1-2)$$

The curvature χ_x of the deformed middle surface is related to the radius of curvature by the relationship $\chi_x = 1/r_x$. Curvature is assumed positive if it is

convex downward. Equation (1-2) can then be written as

$$\varepsilon_x = \chi_x z \quad (1-3)$$

Similarly, in the y-direction,

$$\varepsilon_y = \frac{z}{r_y} \quad (1-4)$$

and

$$\varepsilon_y = \chi_y z \quad (1-5)$$

where,

ε_y = strain in the y-direction

r_y = radius of curvature of the deformed middle surface in the y-direction

χ_y = curvature in the y-direction

The quantity χ_x is related to the deflection w and slope dw/dx by the expression (Shenk 1997):

$$\chi_x = -\frac{d^2 w}{dx^2} \left[1 + \left(\frac{dw}{dx} \right)^2 \right]^{3/2}$$

The expression $d^2 w/dx^2$ in the nominator may be written as $(d/dx)(dw/dx)$. The negative sign in the nominator indicates that the slope dw/dx is decreasing as the location moves away from the origin point o in Fig. 1-3.

For a small deflection w , the square of the slope dw/dx in the denominator is small compared to the quantity 1.0 and can thus be neglected. The above expression becomes

$$\chi_x = -\frac{d^2 w}{dx^2} \quad (1-6)$$

similarly, in the y-direction,

$$\chi_y = -\frac{d^2 w}{dy^2} \quad (1-7)$$

Substituting Eqs. (1-6) and (1-7) into Eqs. (1-3) and (1-5) gives the relationship between strain and deflection

$$\varepsilon_x = -z \frac{d^2 w}{dx^2} \quad (1-8)$$

$$\varepsilon_y = -z \frac{d^2 w}{dy^2} \quad (1-9)$$

The shearing strain-deformation relationship is obtained from Fig. 1-4. If an infinitesimal element of length dx and width dy undergoes shearing deformations,

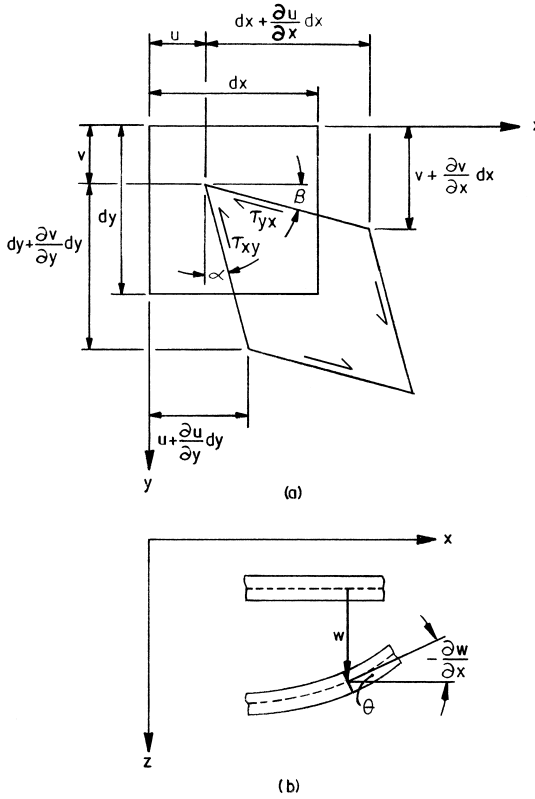


Figure 1-4. Shear deformation of rectangular plate.

α and β , due to in-plane shearing forces and twisting moments then from Fig. 1-4a

$$\sin \alpha \approx \alpha \approx \frac{\frac{\partial u}{\partial y} dy}{\left(1 + \frac{\partial v}{\partial y}\right) dy}$$

or, for small shearing angles,

$$\alpha = \frac{\partial u}{\partial y}.$$

Similarly,

$$\sin \beta \approx \beta \approx \frac{\frac{\partial v}{\partial x} dx}{\left(1 + \frac{\partial u}{\partial x}\right) dx}$$

$$\beta = \frac{\partial v}{\partial x}.$$

Hence,

$$\gamma_{xy} = \alpha + \beta = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (1-10)$$

where

u = deflection in the x -direction;

v = deflection in the y -direction;

γ_{xy} = shearing strain;

$\frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ = shearing strains due to twisting.

The rotation of the middle surface is shown in Fig. 1-4b and is given by $\partial w / \partial x$. Due to this rotation, any point at distance z from the middle surface will deflect by the amount

$$u = z \tan \theta \approx z\theta$$

or

$$u = -z \frac{\partial w}{\partial x} \quad v = -z \frac{\partial w}{\partial y}.$$

Hence, Eq. (1-10) becomes

$$\gamma_{xy} = -2z \frac{\partial^2 w}{\partial x \partial y}. \quad (1-11)$$

Equations (1-8), (1-9), and (1-11) can be written as

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = -z \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \quad (1-12)$$

and are sufficiently accurate for developing the bending theory of thin plates. More precise strain expressions that are a function of the three displacement functions u , v , and w will be derived later when the buckling theory of thin plates is discussed.

1-3 Stress–Deflection Expressions

Equation (1-12) can be expressed in terms of stress rather than strain for ease of calculation. The relationship between stress and strain, excluding thermal loads, in a three-dimensional homogeneous and isotropic element (Fig. 1-5) is obtained

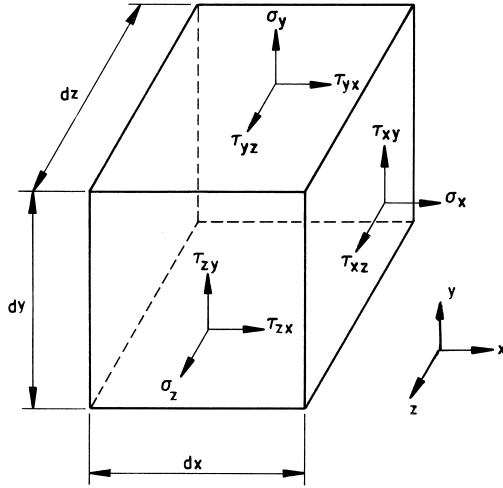


Figure 1-5. Stress Components.

from the theory of elasticity (Sokolnikoff 1956) as

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\mu & -\mu & 0 & 0 & 0 \\ -\mu & 1 & -\mu & 0 & 0 & 0 \\ -\mu & -\mu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\mu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\mu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\mu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} \quad (1-13)$$

where

ϵ = axial strain;

σ = axial stress;

γ = shearing strain;

τ = shearing stress;

E = modulus of elasticity;

μ = Poisson's ratio.

The quantities x , y , and z refer to the directions shown in Fig. 1-5. The quantity $2(1 + \mu)/E$ is usually written as $1/G$ where G is called the shearing modulus.

The stress perpendicular to the surface, i.e., in the z -direction, has a maximum value equal to the applied pressure. For the majority of plate applications in bending, the stress σ_z in the z -direction is small compared to the stress in the other two directions and thus can be neglected. In addition, the shearing stresses τ_{yz} and τ_{zx} are not needed in the formulation of a two-dimensional state of stress. Hence, for this condition, Eq. (1-13) can be written as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}. \quad (1-14)$$

Substituting Eq. (1-12) into Eq. (1-14) gives

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{-Ez}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & (1-\mu) \end{bmatrix} \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}. \quad (1-15)$$

The elastic moduli of elasticity and Poisson's ratio for some commonly used materials are given in Table 1-1. The value of Poisson's ratio is relatively constant

Table 1-1. Moduli of elasticity and Poisson's ratio

Material	Poisson's Ratio	Room Temperature	Modulus of Elasticity ^a Temperature, °F				
			200	400	600	800	1000
Aluminum (6061)	0.33	10.0	9.6	8.7			
Brass (C71000)	0.33	20.0	19.5	18.8	17.8		
Bronze (C61400)	0.33	17.0	16.6	16.0	15.1		
Carbon Steel (C < 0.3)	0.29	29.5	28.8	27.7	26.7	24.2	20.1
Copper (C12300)	0.33	17.0	16.6	16.0	15.1		
Cu–Ni (70–30) (C71500)	0.33	22.0	21.5	20.7	19.6		
Nickel alloy C276	0.29	29.8	29.1	28.3	27.6	26.5	25.3
Nickel alloy 600	0.29	31.0	30.2	29.5	28.7	27.6	26.4
Stainless steel (304)	0.31	28.3	27.6	26.5	25.3	24.1	22.8
Titanium (Gr.1,2)	0.32	15.5	15.0	14.0	12.6	11.2	
Zirconium alloys	0.35	14.4	13.4	11.5	9.9		
Concrete	0.15	3.1 ^b					
Wood, hard		2.1					
Wood, soft		1.3					

^aIn million psi.

^bFor 3000 psi concrete.

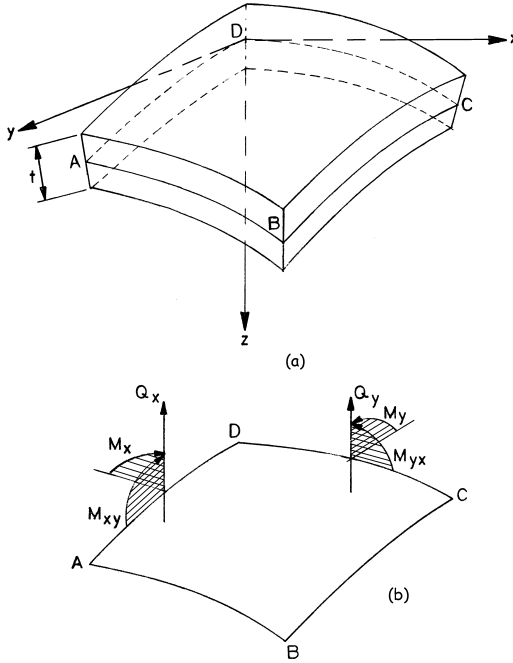


Figure 1-6. Moments in a plate.

at various temperatures for a given material and is thus listed only for room temperature in Table 1-1.

1-4 Force–Stress Expressions

Equation (1-15) can be utilized more readily when the stress values are replaced by moments. This is because the moments at the edges of the plate are needed to satisfy some of the boundary conditions in solving the differential equation. The relationship between moment and stress is obtained from Fig. 1-6a. The moments shown in Fig. 1-6b are positive and are per unit length. By definition, the sum of the moments about the neutral axis due to the internal forces is equal to the sum of the moments of the external forces. Hence,

$$\begin{bmatrix} M_x \\ M_y \\ -M_{xy} \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} z dz. \quad (1-16)$$

The negative sign of M_{xy} in Eq. (1-16) is needed since the direction of M_{xy} in Fig. 1-6b results in a shearing stress τ_{xy} that has a direction opposite to that defined in Fig. 1-4a in the positive z -axis. Substituting Eq. (1-15) into Eq. (1-16) results in

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = -D \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & -(1-\mu) \end{bmatrix} \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \quad (1-17)$$

where

$$D = \frac{Et^3}{12(1-\mu^2)} \quad (1-18)$$

The quantity D is the bending stiffness of a plate. It reduces to the quantity EI , which is the bending stiffness of a beam of unit width, when we let $\mu = 0$.

Problems

1-1 The finite element formulation for the stiffness of a solid three-dimensional element is based on the strain-stress matrix Eq. (1-13). Rewrite this equation as a stress-strain matrix.

1-2 A strain gage rosette is mounted on the flat inside surface of a valve casting. The valve is then pressurized and the following strain values were measured

$$\epsilon_x = 300 \times 10^{-6} \text{ inches/inch}$$

$$\epsilon_y = 150 \times 10^{-6} \text{ inches/inch}$$

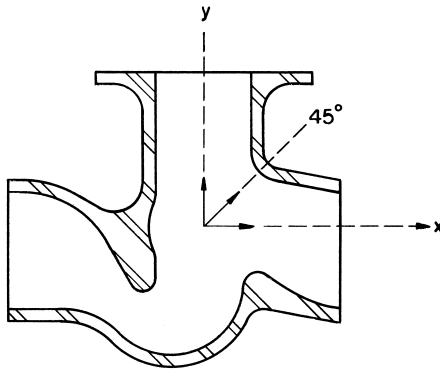
$$\epsilon_{45} = 600 \times 10^{-6} \text{ inches/inch.}$$

Calculate the maximum stresses if $E = 20,000$ ksi and $\mu = 0.15$. *Hint:* first, calculate the shearing strain γ_{xy} at the location of the strain gage from Mohr's circle which is expressed as

$$\epsilon_\theta = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

where $\theta = 45^\circ$ in this case. Then, calculate the principal strains (Beer and Johnson 1981)

$$\epsilon_{\max/\min} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$



Prob. 1-2. Valve body.

$$\frac{1}{2} \gamma_{\max} = \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\theta = \frac{1}{2} \tan^{-1} \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

where 2θ is the orientation of the plane of maximum strain with respect to the plane of given strains. The last step is to use Eq. (1-14) to obtain maximum stress.

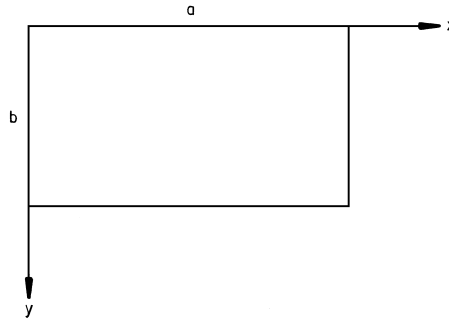
1-3 The stress in the x -direction of a point on the surface of a plate is equal to 30 MPa. The stress in the y -direction is equal to 70 MPa and that in the z -direction is equal to -0.6 MPa. Determine the maximum shearing stress by Mohr's circle and show the plane on which it acts.

1-4 Determine the maximum bending stress values σ_x and σ_y in a simply supported plate, with length $a = 100$ cm and width $b = 75$ cm. The deflection is approximated by

$$w = k \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

where k is a constant equal to 0.462 cm. Let $t = 1.2$ cm, $\mu = 0.3$, and $E = 200,000$ MPa.

1-5 A simply supported rectangular plate with dimensions $a = 30$ inches and $b = 20$ inches is subjected to a uniform pressure of 15 psi. Determine the maximum bending moment in the middle of the plate by taking unit strips in the middle of the plate in the x - and y -directions. Assume the strips to be connected at point A . Compare the results with the more accurate solution obtained from the plate theory in Example 1-2.



Prob. 1-4. Simply supported plate.

1-5 Governing Differential Equations

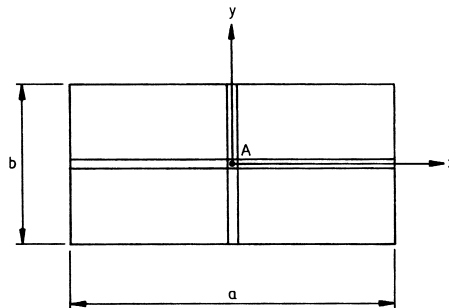
The governing differential equation of a beam in bending is

$$\frac{d^2 w}{dx^2} = -\frac{M(x)}{EI} \quad (1-19)$$

which can be expressed in terms of applied loads as

$$\frac{d^4 w}{dx^4} = \frac{p(x)}{EI}. \quad (1-20)$$

A similar equation can be written for the bending of a plate. The corresponding differential equation for the bending of a plate is more complicated because it must include terms for the bending in the x - and y -directions as well as torsional moments that are present in the plate. Lagrange (Timoshenko 1983) was the first to develop the differential equation for the bending of a rectangular plate in 1811. We begin the derivation of the governing equations by considering an infinitesimal element dx, dy in Fig. 1-7 subjected to lateral



Prob. 1-5. Simply supported plate with unit strips.

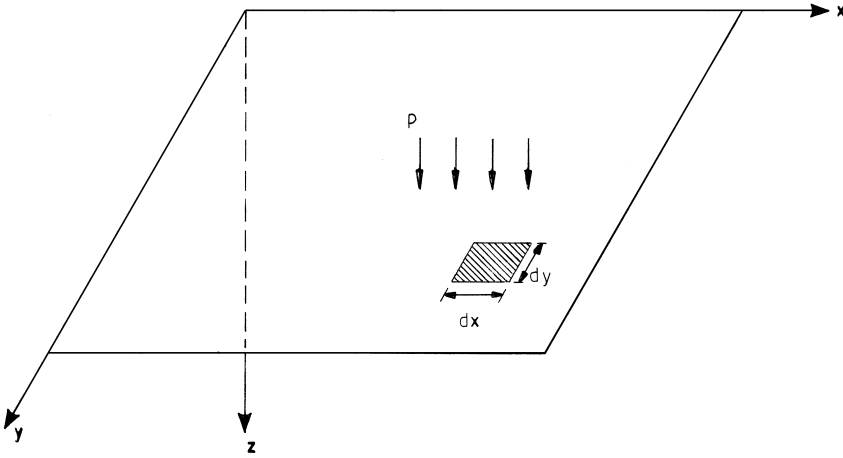


Figure 1-7. Lateral loads on a rectangular plate.

loads p . The forces and moments, per unit length, needed for equilibrium are shown in Fig. 1-8 and are positive as shown. Also, downward deflection is taken as positive. It is of interest to note that two shearing forces, Q_x and Q_y , and two torsional moments, M_{xy} and M_{yx} , are needed to properly define the equilibrium of a rectangular plate.

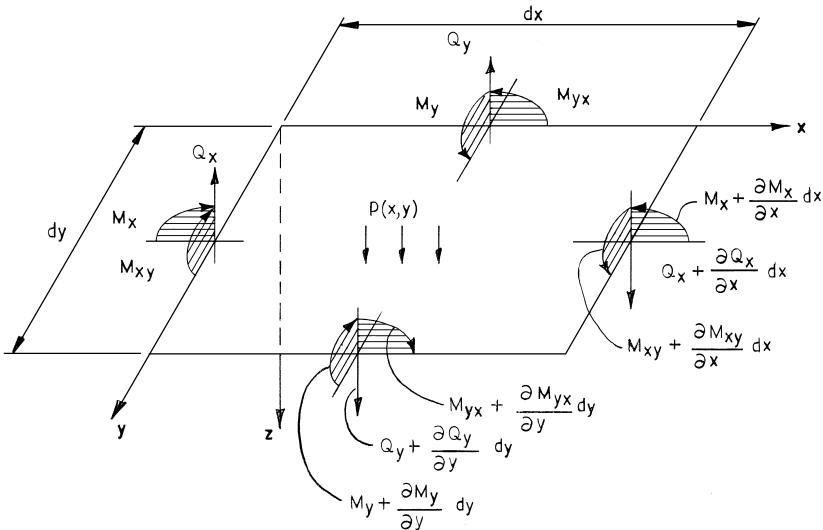


Figure 1-8. Forces and Moments in a rectangular plate.

Summation of forces in the z -direction gives the first equation of equilibrium:

$$p(x,y)dx dy - Q_x dy + \left(Q_x + \frac{\partial Q_x}{\partial x} dx \right) dy - Q_y dx + \left(Q_y + \frac{\partial Q_y}{\partial y} dy \right) dx = 0.$$

This equation reduces to

$$p(x,y) + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0. \quad (1-21)$$

Summation of moments around the x -axis gives the second equation of equilibrium

$$\begin{aligned} M_y dx - \left(M_y + \frac{\partial M_y}{\partial y} dy \right) dx - M_{xy} dy + \left(M_{xy} + \frac{\partial M_{xy}}{\partial x} dx \right) dy \\ + \left(Q_y + \frac{\partial Q_y}{\partial y} dy \right) dx dy - Q_x dy dy/2 \\ + \left(Q_x + \frac{\partial Q_x}{\partial x} dx \right) dy dy/2 + p dx dy dy/2 = 0. \end{aligned}$$

Simplifying this equation gives

$$Q_y + \frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + \left(\frac{\partial Q_y}{\partial y} + \frac{1}{2} \frac{\partial Q_x}{\partial x} + \frac{1}{2} p \right) dy = 0.$$

The bracketed term in this equation is multiplied by an infinitesimal quantity dy . It can thus be deleted because its magnitude is substantially less than that of the other three terms. The equation becomes

$$\frac{\partial Q_y}{\partial x} = \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial x \partial y}. \quad (1-22)$$

Summation of moments around the y -axis gives the third equation of equilibrium

$$\frac{\partial Q_x}{\partial y} = \frac{\partial^2 M_x}{\partial x^2} - \frac{\partial^2 M_{yx}}{\partial x \partial y}. \quad (1-23)$$

Substituting Eqs. (1-22) and (1-23) into Eq. (1-21) gives

$$p(x,y) + \frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = 0 \quad (1-24)$$

In this equation it is assumed that $M_{xy} = M_{yx}$ because at any point on the plate the shearing stress $\tau_{xy} = -\tau_{yx}$.

Substituting Eq. (1-17) into this equation gives

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = p(x,y)/D \quad (1-25)$$

A comparison of this equation with Eq. (1-20) for the bending of beams indicates that Eq. (1-25) is considerably more complicated because it considers the deflection in the x - and y -directions as well as the shearing effects in the xy plane.

Equation (1-25) can also be written as

$$\nabla^2 \nabla^2 w = \nabla^4 w = p(x, y)/D \quad (1-26)$$

where

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

and

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}$$

Equation (1-26) is the basic differential equation for rectangular plates in bending. A solution of this equation yields an expression for the deflection, w , of the plate. The moment expressions are obtained by substituting the deflection expressions into Eq. (1-17). The shear forces are obtained from Eqs. (1-22), (1-23), and (1-17), and are given by

$$Q_x = -D \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) \quad (1-27)$$

$$Q_y = -D \left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} \right) \quad (1-28)$$

For sign convention we will assume a downward deflection as positive in Eq. (1-26). All other quantities are assumed positive as shown in Fig. 1-8.

Problems

1-6 Find M_x , M_y , M_{xy} , Q_x , and Q_y of a rectangular plate whose deflection is given by

$$w = k \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

where k , a , b , n , and m are constants.

1-7 Derive Eqs. (1-25), (1-27), and (1-28).

1-6 Boundary Conditions

The most frequently encountered boundary conditions for rectangular plates are essentially the same as those for beams. They are either fixed, simply supported, free, or partially fixed as shown in Fig. 1-9.

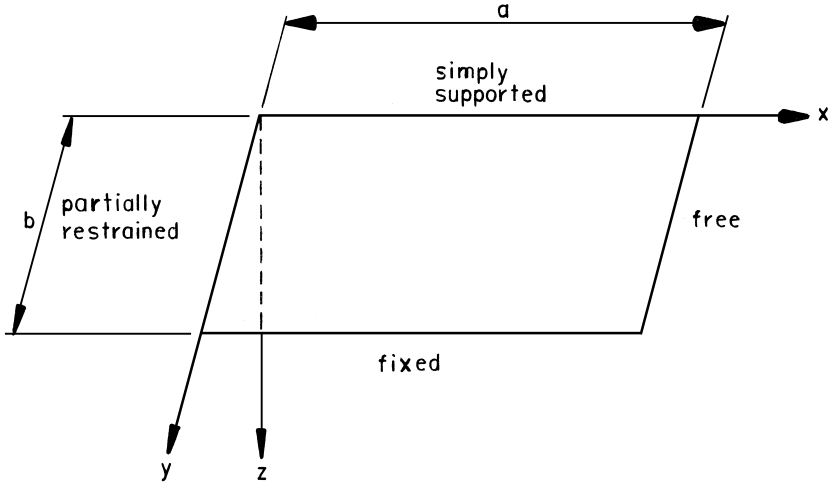


Figure 1-9. Plate with various boundary conditions.

(a) Fixed Edges: For a fixed edge (Fig. 1-9), the deflection and slope are zero. Thus,

$$w|_{y=b} = 0 \quad (1-29)$$

$$\left. \frac{\partial w}{\partial y} \right|_{y=b} = 0. \quad (1-30)$$

(b) Simply Supported Edge: For a simply supported edge (Fig. 1-9), the deflection and moment are zero. Hence,

$$w|_{y=0} = 0 \quad (1-31)$$

and, from Eq. (1-17),

$$M_y|_{y=0} = -D \left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} \right) \bigg|_{y=0} = 0. \quad (1-32)$$

The expression $\mu \frac{\partial^2 w}{\partial x^2}$ in Eq. (1-32) can be written as $\mu \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x} \right)$ which is the rate of change of the slope at the boundary. But the change in slope along the simply supported edge $y = 0$ is always zero. Hence the quantity $\mu \frac{\partial^2 w}{\partial x^2}$ vanishes and the moment boundary condition becomes

$$M_y|_{y=0} = \frac{\partial^2 w}{\partial y^2} \bigg|_{y=0} = 0. \quad (1-33)$$

(c) Free Edge: At a free edge, the moment and shear are zero. Hence,

$$M_x|_{x=a} = M_{xy}|_{x=a} = Q_x|_{x=a} = 0.$$

From the first of these boundary conditions and Eq. (1-17) we get

$$\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right) \Big|_{x=a} = 0. \quad (1-34)$$

The other two boundary conditions can be combined into a single expression. Referring to Fig. 1-10, it was shown by Kirchhoff (Timoshenko and Woinowsky-Krieger 1959) that the moment M_{xy} can be thought of as a series of couples acting on an infinitesimal section. Hence, at any point along the edge

$$Q' = - \left(\frac{\partial M_{xy}}{\partial y} \right) \Big|_{x=a}$$

This equivalent shearing force, Q' , must be added to the shearing force Q_x acting at the edge. Hence the total shearing force at the free edge is given by Q' and Eq. (1-27) as

$$V_x = \left(Q_x - \frac{\partial M_{xy}}{\partial y} \right) \Big|_{x=a} = 0$$

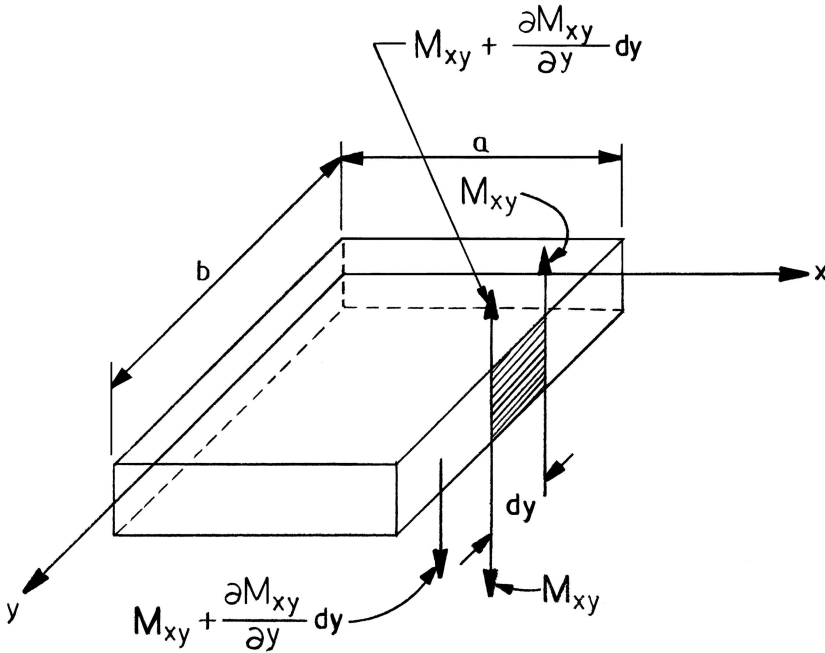


Figure 1-10. Edge shear.

Substituting the values of Q_x and M_{xy} from Eqs. (1-27) and (1-17) into this equation gives

$$\left(\frac{\partial^3 w}{\partial x^3} + (2 - \mu) \frac{\partial^3 w}{\partial x \partial y^2} \right) \bigg|_{x=a} = 0. \quad (1-35)$$

Equations (1-34) and (1-35) are the two necessary boundary conditions at a free edge of a rectangular plate.

(d) Partially Fixed Edge: A partially fixed edge occurs in continuous plates or plates connected to beams. For this latter condition, Fig. 1-11 shows that the two boundary conditions are given by

$$V|_{\text{plate}} = V|_{\text{beam}}$$

$$D \left[\frac{\partial^3 w}{\partial x^3} + (2 - \mu) \frac{\partial^3 w}{\partial x \partial y^2} \right] \bigg|_{x=0} = EI \left(\frac{\partial^4 w}{\partial y^4} \right) \bigg|_{x=0} \quad (1-36)$$

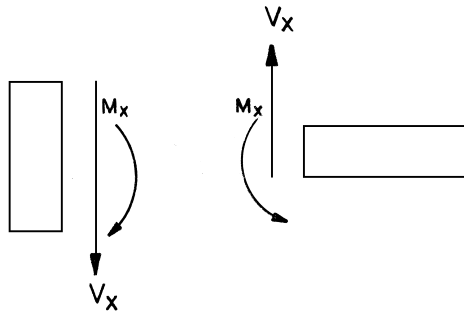
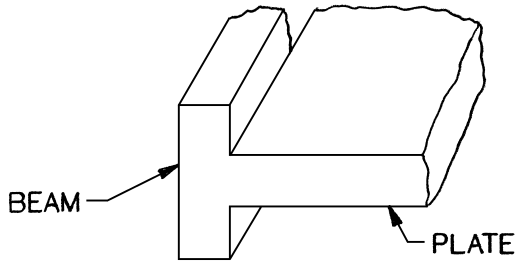


Figure 1-11. Plate with edge beam support.

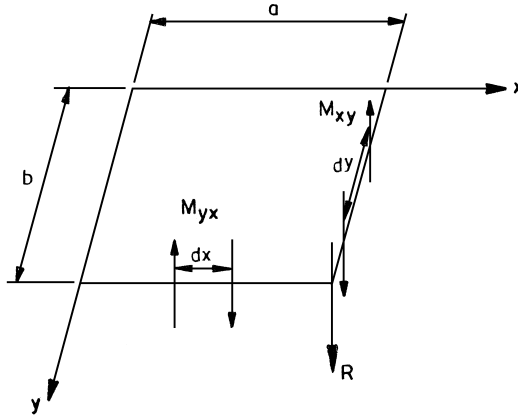


Figure 1-12. Edge shear moments.

and

$$M|_{\text{plate}} = M|_{\text{beam}}$$

$$D \left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right) \Big|_{x=0} = GJ \left(\frac{\partial^3 w}{\partial x \partial y^2} \right) \Big|_{x=0} \quad (1-37)$$

(e) Corner Reactions: It was shown in the derivation of Eq. (1-35) that the torsion moment M_{xy} shown in Fig. 1-10 can be resolved into a series of couples. At any corner, say $x = a$ and $y = b$ in Fig. 1-12, the moment M_{xy} results in a downward force and so does M_{yx} as shown in the figure. Hence the total reaction at $x = a$ and $y = b$ is given by

$$R = 2(M_{xy}) \Big|_{\substack{x=a \\ y=b}} = 2D(1 - \mu) \left(\frac{\partial^2 w}{\partial x \partial y} \right) \Big|_{\substack{x=a \\ y=b}} \quad (1-38)$$

Equation (1-38) is normally used to determine the force in corner bolts of rectangular cover plates of gear transmission casings, flanges, etc.

To summarize, Eqs. (29) and (30) are used for fixed edges whereas Eqs. (31) and (33) are utilized for simply supported edges. Free edges are expressed by Eqs. (34) and (35) and boundaries of plates with beam edges are given by Eqs. (36) and (37). Corner loads are expressed by Eq. (1-38).

Example 1-1

Find the moment and reaction expressions for a simply supported rectangular plate (Fig. 1-13a) of length a , width b , and subjected to a sinusoidal load given by

$$p = p_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}.$$

Solution

The differential Eq. (1-26) is written as

$$\nabla^4 w = \frac{p_0}{D} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}. \quad (1)$$

From Fig. 1-13, the boundary conditions are

$$w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = a$$

$$w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad y = b.$$

The assumed expression for the deflection must be of the same general format as that of the applied load in order to solve Eq. (1). It must also satisfy the

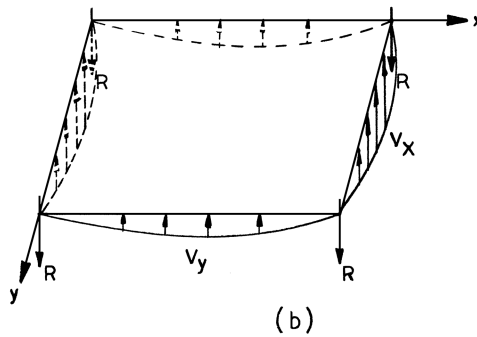
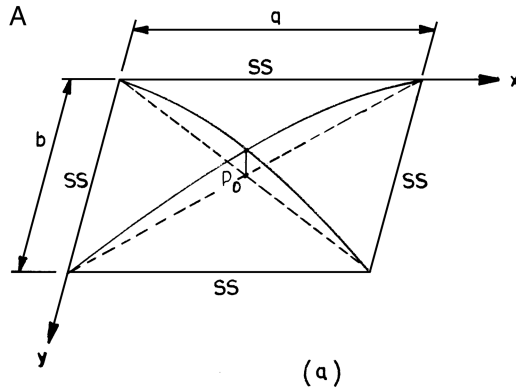
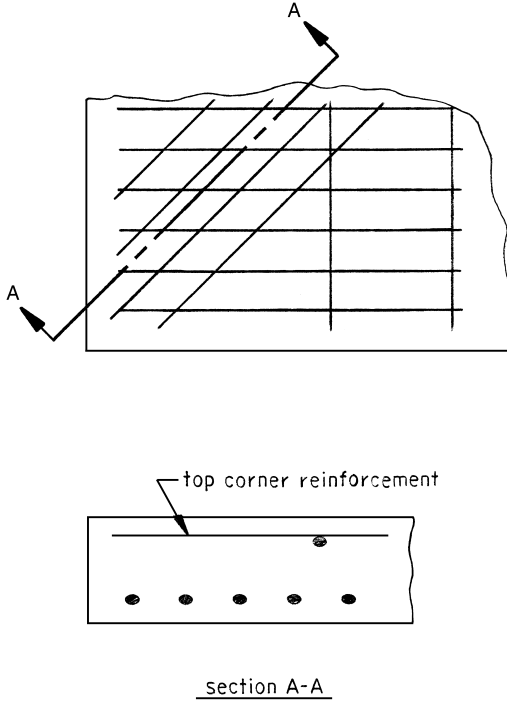


Figure 1-13a and b. Sinusoidally loaded plate.

**Figure 1-13c.** Corner reinforcement.

boundary conditions of the plate. Hence, a deflection of the form

$$w = C \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \quad (2)$$

satisfies the given boundary conditions.

Substituting Eq. (2) into Eq. (1) gives

$$C = \frac{P_o}{D\pi^4 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2}$$

and the expression for w becomes

$$w = \frac{P_o}{D\pi^4 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}. \quad (3)$$

Substituting this expression into Eq. (1-17) gives

$$M_x = \frac{P_o}{\pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2} \left(\frac{1}{a^2} + \frac{\mu}{b^2} \right) \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

$$M_y = \frac{p_o}{\pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2} \left(\frac{\mu}{a^2} + \frac{1}{b^2} \right) \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

$$M_{xy} = \frac{p_o(1-\mu)}{\pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2} \cos \frac{\pi x}{a} \cos \frac{\pi y}{b}.$$

The maximum value for moments M_x and M_y occurs at $x = a/2$ and $y = b/2$.

To find the reactions, we calculate Q_x and Q_y from Eqs. (1-27) and (1-28). This gives

$$Q_x = \frac{p_o}{\pi a \left(\frac{1}{a^2} + \frac{1}{b^2} \right)} \cos \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

$$Q_y = \frac{p_o}{\pi b \left(\frac{1}{a^2} + \frac{1}{b^2} \right)} \sin \frac{\pi x}{a} \cos \frac{\pi y}{b}$$

for edge $x = a$, the reaction is given by

$$V_x = \left(Q_x - \frac{\partial M_{xy}}{\partial y} \right) \Big|_{x=a} = \frac{-p_o}{\pi a \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2} \left(\frac{1}{a^2} + \frac{2-\mu}{b^2} \right) \sin \frac{\pi y}{b} \quad (4)$$

and for edge $y = b$, the reaction is given by

$$V_y = \left(Q_y - \frac{\partial M_{xy}}{\partial x} \right) \Big|_{y=b} = \frac{-p_o}{\pi b \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2} \left(\frac{1}{b^2} + \frac{2-\mu}{a^2} \right) \sin \frac{\pi x}{a}. \quad (5)$$

The total reaction around the plate is obtained by integrating Eqs. (4) and (5) from $x = 0$ to $x = a$ and from $y = 0$ to $y = b$ and then multiplying the result by 2 due to symmetry. This gives

$$\text{total reaction} = \frac{4p_o ab}{\pi^2} + \frac{8p_o(1-\mu)}{\pi^2 ab \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2}. \quad (6)$$

The first part of this equation can also be obtained by integrating the applied load over the total area, or

$$\int_0^b \int_0^a p \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} dx dy.$$

The second expression in Eq. (6) is the summation of the four corner reactions given by Eq. (1-38). Hence, at $x = 0$ and $y = 0$ the expression for the reaction is

$$R = 2(M_{xy}) \Big|_{x=0, y=0} = \frac{2p_o(1-\mu)}{\pi^2 ab \left(\frac{1}{a^2} + \frac{1}{b^2} \right)^2}.$$

A plot of the shear distribution and reaction is shown in Fig. 1-13b. The positive value of R indicates that the corners have a tendency to lift up and a downward force is needed to keep them in place. This action must be considered when designing cover plates and concrete slabs. An example of the reinforcement at the corners of a concrete slab is shown in Fig. 1-13c.

1-7 Double Series Solution of Simply Supported Plates

The first successful solution of a simply supported rectangular plate subjected to uniform load was made by Navier (Timoshenko 1983) in 1820. He assumed the load p in Eq. (1-26) to be represented by the double Fourier series, Appendix A, of the form

$$p(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (1-39)$$

where p_{mn} is obtained from

$$p_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \quad (1-40)$$

and $f(x,y)$ is the shape of the applied load.

Similarly the deflection w is expressed by

$$w(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \quad (1-41)$$

This equation automatically satisfies four boundary conditions of a simply supported plate and w_{mn} is a constant that is determined from the differential equation.

The solution of a rectangular plate problem consists of obtaining a load function from Eq. (1-39). The unknown constant w_{mn} is obtained by substituting Eqs. (1-39) and (1-41) into Eq. (1-26).

Example 1-2

- (a) Determine the maximum bending moment of a simply supported plate due to a uniformly applied load.
- (b) Let a steel rectangular plate with dimensions $a = 30$ inch and $b = 20$ inch be subjected to a pressure of 15 psi. Determine the maximum bending moment and deflection if $\mu = 0.3$, $E = 30,000$ ksi, and $t = 0.38$ inch.

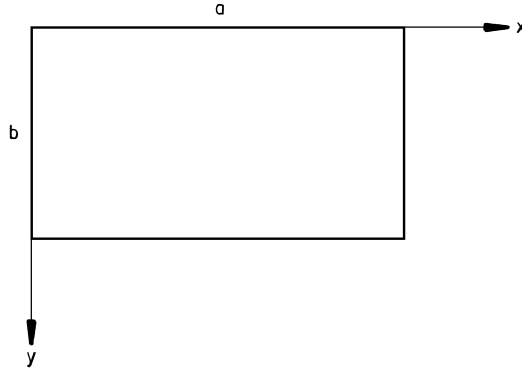


Figure 1-14. Uniformly loaded plate.

Solution

(a) Let the coordinate system be as shown in Fig. 1-14. Equation (1-40) can be solved by letting $f(x, y)$ equal a constant p_0 because the load is uniform over the entire plate. Hence,

$$p_{mn} = \frac{4p_0}{ab} \int_0^b \int_0^a \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy = \frac{4p_0}{\pi^2 mn} (\cos m\pi - 1)(\cos n\pi - 1) = \frac{16p_0}{\pi^2 mn}$$

where

$$\begin{aligned} m &= 1, 3, 5, \dots \\ n &= 1, 3, 5, \dots \end{aligned}$$

From Eq. (1-39),

$$p = \frac{16p_0}{\pi^2 mn} \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}.$$

Substituting this equation and Eq. (1-41) into Eq. (1-26) gives

$$w_{mn} = \frac{16p_0}{\pi^6 mn D [(m/a)^2 + (n/b)^2]^2} \quad \begin{aligned} m &= 1, 3, 5, \dots \\ n &= 1, 3, 5, \dots \end{aligned}$$

Hence, the deflection expression becomes

$$w = \frac{16p_0}{\pi^6 D} \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} \frac{\sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}}{mn [(m/a)^2 + (n/b)^2]^2}. \quad (1)$$

The bending and torsional moment expressions are given by Eq. (1-17) and are expressed as

$$M_x = \frac{16p_0}{\pi^4} \left[\sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} F_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right] \quad (2)$$

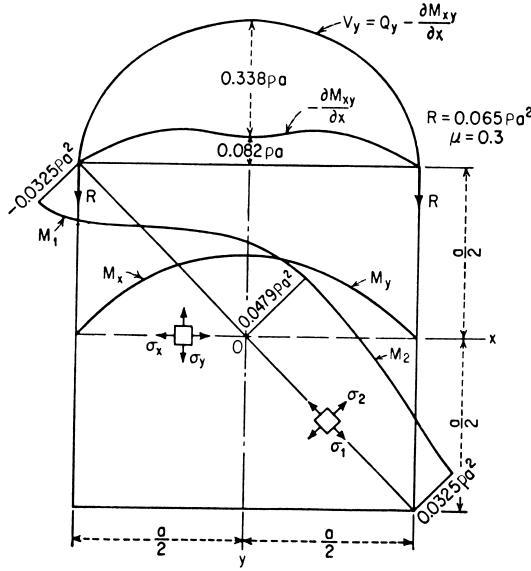


Figure 1-15. Shear and Moment distribution in a uniformly loaded square plate. (Timoshenko and Woinowsky-Krieger 1959.)

$$M_y = \frac{16p_0}{\pi^4} \left[\sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} G_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right] \quad (3)$$

$$M_{xy} = \frac{16p_0(1-\mu)}{\pi^4} \left[\sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} H_{mn} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right] \quad (4)$$

where

$$F_{mn} = \frac{(m/a)^2 + \mu(n/b)^2}{mn[(m/a)^2 + (n/b)^2]^2}$$

$$G_{mn} = \frac{\mu(m/a)^2 + (n/b)^2}{mn[(m/a)^2 + (n/b)^2]^2}$$

$$H_{mn} = \frac{1}{ab[(m/a)^2 + (n/b)^2]^2}.$$

The maximum values of deflection and bending moments occur at $x = a/2$ and $y = b/2$.

A plot of Eqs. (2), (3), and (4) is shown in Fig. 1-15 for a square plate. The figure also shows a plot of M_1 and M_2 obtained from Mohr's circle along the diagonal of the plate. M_1 becomes negative near the corner of the plate. This is due to the uplift tendency at the corners. This uplift is resisted by the reaction R that causes tension at the top portion of the plate near the corners. This tension must be properly reinforced in concrete slabs as shown in Fig. 1-13c.

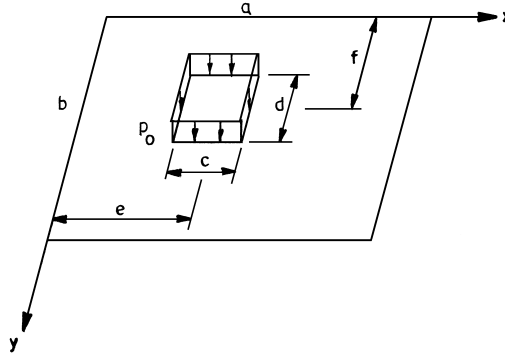


Figure 1-16. Partially loaded plate.

(b) The maximum values of M_x , M_y , and w are obtained from expression (2), (3), and (4) above. The computer program “DBLSUM” in Table A-2 of Appendix A is used to determine the series summation for m and n .

$$M_x = \frac{16p_o b^2}{\pi^4} (0.3035) = 299.1 \text{ inch-lbs/inch}$$

$$M_y = \frac{16p_o b^2}{\pi^4} (0.4941) = 487.0 \text{ inch-lbs/inch}$$

$$w = \frac{16p_o b^4}{\pi^6 D} (0.4647)$$

The value of D is given by Eq. (1-18) as

$$D = \frac{Et^3}{12(1 - \mu^2)} = \frac{30,000,000 \times 0.38^3}{12(0.91)} = 150,747 \text{ lbs-inch.}$$

the maximum deflection is

$$w = 0.12 \text{ inch.}$$

Example 1-3

Find the deflection expression for the simply supported plate loaded as shown in Fig. 1-16.

Solution

The Fourier expansion of the load is obtained from Eqs. (1-39) and (1-40) as

$$p_{mn} = \frac{4}{abcd} \int_{f-d/2}^{f+d/2} \int_{e-c/2}^{e+c/2} p_o \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy$$

$$p_{mn} = \frac{16p_o}{mncd\pi^2} \sin \frac{m\pi e}{a} \sin \frac{m\pi c}{2a} \sin \frac{n\pi f}{b} \sin \frac{n\pi d}{2b}. \quad (1)$$

Substituting this expression and Eq. (1-41) into Eq. (1-26) gives

$$w_{mn} = \frac{16p_o}{\pi^6 cdD} \frac{\sin \frac{m\pi e}{a} \sin \frac{m\pi c}{2a} \sin \frac{n\pi f}{b} \sin \frac{n\pi d}{2b}}{mn[(m/a)^2 + (n/b)^2]^2} \quad (2)$$

and

$$w = \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \quad (3)$$

This equation reduces to Eq. (1) of Example 1-2 for a uniformly loaded plate when $c = a$, $d = b$, $e = a/2$, and $f = b/2$.

Problems

1-8 Find the expression for moments M_x and M_y in Example 1-3.

1-9 A tabletop is loaded as shown in Fig. 1-16. Find the maximum stress in the table if $a = 200$ cm, $b = 80$ cm, $c = 30$ cm, $d = 15$ cm, $e = 80$ cm, $f = 40$ cm, $E = 840$ kg/mm², $p = 0.2$ kg/cm², and $\mu = 0.30$. Assume the tabletop to be simply supported. Let $t = 2$ cm and let $m = 1, 3$ and $n = 1, 3$.

1-10 Show that in Example 1-3 the value of p_{mn} for a concentrated load, p_o , is

$$p_{mn} = \frac{4p_o}{ab} \sin \frac{m\pi e}{a} \sin \frac{m\pi f}{b}.$$

1-8 Single Series Solution of Simply Supported Plates

Levy (Timoshenko 1983) in 1900 developed a method for solving simply supported plates subjected to various loading conditions using single Fourier series. This method is more practical than Navier's solution because it can also be used in plates with various boundary conditions as discussed in Chapter 2. Levy suggested the solution of Eq. (1-26) to be expressed in terms of homogeneous and particular parts each of which consists of a single Fourier series where the unknown function is determined from the boundary conditions. The solution is expressed as

$$w = w_h + w_p. \quad (1-42)$$

The homogeneous solution is written as

$$w_h = \sum_{m=1}^{\infty} f_m(y) \sin \frac{m\pi x}{a} \quad (1-43)$$

where $f(y)$ indicates that it is a function of y only. This equation also satisfies a simply supported boundary condition at $x = 0$ and $x = a$. Substituting Eq. (1-43) into the differential equation

$$\nabla^4 w = 0$$

gives

$$\left[\left(\frac{m\pi}{a} \right)^4 f_m(y) - 2 \left(\frac{m\pi}{a} \right)^2 \frac{d^2 f_m(y)}{dy^2} + \frac{d^4 f_m(y)}{dy^4} \right] \sin \frac{m\pi x}{a} = 0$$

which is satisfied when the bracketed term is equal to zero. Thus,

$$\frac{d^4 f_m(y)}{dy^4} - 2 \left(\frac{m\pi}{a} \right)^2 \frac{d^2 f_m(y)}{dy^2} + \left(\frac{m\pi}{a} \right)^4 f_m(y) = 0. \quad (1-44)$$

The solution of this differential equation can be expressed as

$$f_m(y) = F_m e^{R_m y}. \quad (1-45)$$

Substituting Eq. (1-45) into Eq. (1-44) gives

$$R_m^4 - 2 \left(\frac{m\pi}{a} \right)^2 R_m^2 + \left(\frac{m\pi}{a} \right)^4 = 0$$

which has the roots

$$R_m = \pm \frac{m\pi}{a}, \quad \pm \frac{m\pi}{a}.$$

Thus, the solution of Eq. (1-44) is

$$f_m(y) = C_{1m} e^{\frac{m\pi y}{a}} + C_{2m} e^{-\frac{m\pi y}{a}} + C_{3m} y e^{\frac{m\pi y}{a}} + C_{4m} y e^{-\frac{m\pi y}{a}}$$

where C_{1m} , C_{2m} , C_{3m} , and C_{4m} are constants. This equation can also be written as

$$f_m(y) = A_m \sinh \frac{m\pi y}{a} + B_m \cosh \frac{m\pi y}{a} + C_m y \sinh \frac{m\pi y}{a} + D_m y \cosh \frac{m\pi y}{a}.$$

The homogeneous solution given by Eq. (1-43) becomes

$$w_h = \sum_{m=1}^{\infty} \left[A_m \sinh \frac{m\pi y}{a} + B_m \cosh \frac{m\pi y}{a} + C_m y \sinh \frac{m\pi y}{a} + D_m y \cosh \frac{m\pi y}{a} \right] \sin \frac{m\pi x}{a} \quad (1-46)$$

where the constants A_m , B_m , C_m , and D_m are obtained from the boundary conditions.

The particular solution, w_p , in Eq. (1-42) can be expressed by a single Fourier series as

$$w_p = \sum_{m=1}^{\infty} k_m(y) \sin \frac{m\pi x}{a}. \quad (1-47)$$

The load p is expressed as

$$p(x, y) = \sum_{m=1}^{\infty} p_m(y) \sin \frac{m\pi x}{a} \quad (1-48)$$

where

$$p_m(y) = \frac{2}{a} \int_0^a p(x, y) \sin \frac{m\pi x}{a} dx. \quad (1-49)$$

Substituting Eqs. (1-47) and (1-48) into Eq. (1-26) gives

$$\frac{d^4 k_m}{dy^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 k_m}{dy^2} + \left(\frac{m\pi}{a}\right)^4 k_m = \frac{p_m(y)}{D}. \quad (1-50)$$

Thus, the solution of the differential equation (1-26) consists of solving Eqs. (1-46) and (1-50) as shown in the following example.

Example 1-4

The rectangular titanium plate shown in Fig. 1-17 is subjected to a uniform load p_o . Determine the expression for the deflection.

Solution

From Eq. (1-49),

$$p_m(y) = \frac{2p_o}{a} \int_0^a \sin \frac{m\pi x}{a} dx = \frac{2p_o}{m\pi} (\cos m\pi - 1) = \frac{4p_o}{m\pi} \quad m = 1, 3, \dots$$

Hence, Eq. (1-50) becomes

$$\frac{d^4 k_m}{dy^4} - 2\left(\frac{m\pi}{a}\right)^2 \frac{d^2 k_m}{dy^2} + \left(\frac{m\pi}{a}\right)^4 k_m = \frac{4p_o}{m\pi D}. \quad (1)$$

The particular solution of this equation can be taken as

$$k_m = C.$$

Substituting this expression into Eq. (1-1) gives

$$k_m = \frac{4a^4 p_o}{m^5 \pi^5 D} \quad m = 1, 3, \dots$$

And Eq. (1-47) for the particular solution becomes

$$w_p = \frac{4a^4 p_o}{\pi^5 D} \sum_{m=1,3,\dots}^{\infty} \frac{1}{m^5} \sin \frac{m\pi x}{a}. \quad (2)$$

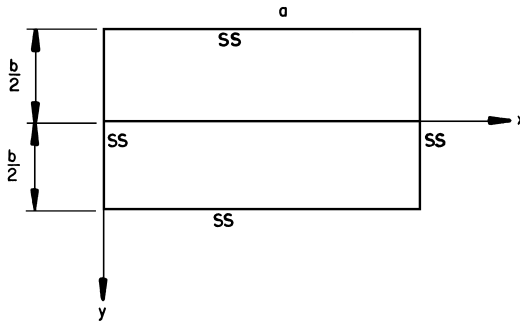


Figure 1-17. Titanium plate.

The homogeneous solution for the deflection is obtained from Eq. (1-46). Referring to Fig. 1-17, the deflection in the y -direction due to uniform load is symmetric about the x -axis. Hence, the constants A_m and D_m must be set to zero since the quantities $\sinh \frac{m\pi y}{a}$ and $y \cosh \frac{m\pi y}{a}$ are odd functions as y varies from positive to negative. Also, m must be set to 1, 3, 5, etc. in order for $\sin \frac{m\pi x}{a}$ to be symmetric around $x = a/2$. Hence,

$$w_h = \sum_{m=1,3,\dots}^{\infty} \left(B_m \cosh \frac{m\pi y}{a} + C_m y \sinh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a}$$

and the total deflection can now be expressed as

$$w = \sum_{m=1,3,\dots}^{\infty} \left(B_m \cosh \frac{m\pi y}{a} + C_m \sinh \frac{m\pi y}{a} + \frac{4p_o a^4}{m^5 \pi^5 D} \right) \sin \frac{m\pi x}{a}. \quad (3)$$

The boundary conditions along the y -axis are expressed as

$$w = 0 \quad \text{at} \quad y = \pm b/2$$

and

$$\frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at} \quad y = \pm b/2.$$

From the first of these boundary conditions we get

$$B_m \cosh \frac{m\pi b}{2a} + C_m \frac{b}{2} \sinh \frac{m\pi b}{2a} + \frac{4a^4 p_o}{m^5 \pi^5 D} = 0$$

and from the second boundary condition we obtain

$$\left[B_m \left(\frac{m\pi}{a} \right) + bc_m \right] \cosh \frac{m\pi b}{2a} + C_m \left(\frac{m\pi b}{2a} \right) \sinh \frac{m\pi b}{2a} = 0.$$

Solving these two simultaneous equations yields

$$C_m = \frac{2a^3 p_o}{m^4 \pi^4 D \cosh \frac{m\pi b}{2a}}$$

and

$$B_m = \frac{4a^4 p_o + m\pi p_o a^3 b \tanh \frac{m\pi b}{2a}}{m^5 \pi^5 D \cosh \frac{m\pi b}{2a}}.$$

With these two expressions known, Eq. (3) can now be solved for various values of x and y .

The load in the previous few examples was assumed uniform in distribution. Other distributions can be used in Eq. (1-49) as long as they can be expressed in

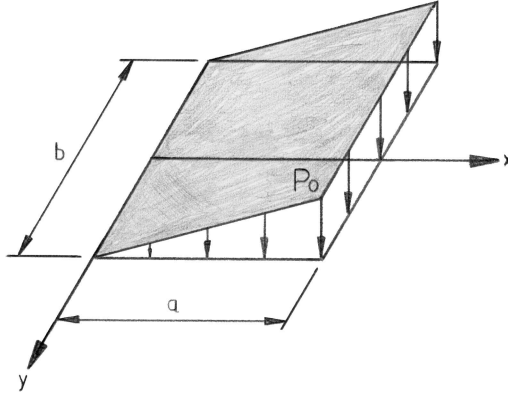


Figure 1-18. Plate with triangular load.

terms of x and y . Thus, if the load in Example 1-4 is triangular in distribution as shown in Fig. 1-18, then it can be expressed as

$$p_m = \frac{p_0 x}{a}$$

and Eq. (1-49) becomes

$$p_m(y) = \frac{2}{a} \int_0^a \frac{p_0 x}{a} \sin \frac{m\pi x}{a} dx = \frac{2p_0}{m\pi} (-1)^{m+1} \quad m = 1, 2, \dots$$

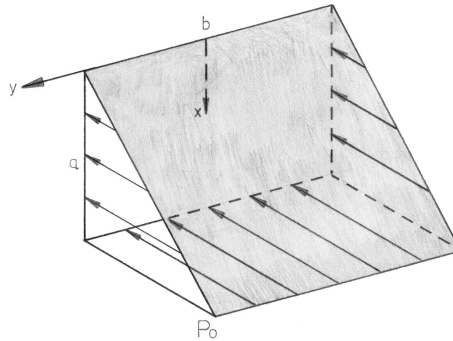
Problems

- 1-11 Find the expression for M_x , M_y and M_{xy} in Example 1-4.
- 1-12 A channel weir is approximated as a simply supported rectangular plate and is subjected to triangular loading. Find the expression for the moments.
- 1-13 An internal zirconium bulkhead is loaded as shown. Find the expression for the moments assuming the plate to be simply supported.

1-9 Design of Rectangular Plates

The procedure for designing a rectangular plate with a given boundary condition and applied lateral loads is

1. Express the loads in a Fourier series and define a similar expression for the deflection.
2. Solve Eq. (1-26) for the actual deflection, w , by utilizing the boundary conditions in the solution.
3. Determine the maximum moments from Eq. (1-17).

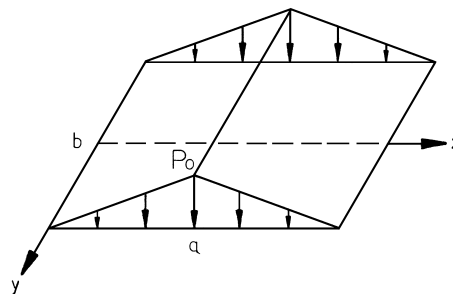


Prob. 1-12. Plate with hydrostatic load.

4. For metallic plates, the required thickness is calculated from the expression $\sigma = Mc/I$ where, for a unit width, it reduces to

$$t = \sqrt{6M/\sigma}. \quad (1-51)$$

5. For reinforced concrete slabs where the reinforcement is about the same in the x - and y -axes, the design can be approximated by Eq. (1-26) as discussed above. The design of concrete plates is more complicated than that of metallic plates because the engineer has to determine not only the magnitude of the bending moments, but also their direction and location in order to properly space the reinforcing bars. Standards such as the ACI 318 establish minimum requirements for concrete thickness and reinforcement spacing throughout the slab.
6. Orthotropic plates and reinforced concrete slabs where the reinforcement is not the same in the x - and y -directions are analyzed in accordance with the orthotropic plate theory discussed in Chapter 5.
7. Plates with variable thickness are solved by modifying the moment expressions in Eq. (1-17) and the deflection terms in Eq. (1-26) by expressing the D term as a function of the variable thickness t (Ventsel and Kautthammer



Prob. 1-13. Triangular load.

2001). Unfortunately, the resulting classical equations are too difficult to express in a closed form for most boundary conditions. Numerical solutions such as the finite difference method discussed in subsequent chapters are best suited for solving such problems.

Allowable stress values at various temperatures of various materials are published in many international codes. The ASME VIII-1 code publishes allowable stresses for over 500 different steels and nonferrous alloys used in pressure vessels. These allowable stresses are obtained by taking the smaller of 2/3 of the yield stress or 1/3.5 of the tensile strength of the material at temperatures where creep and rupture of the material are not a controlling factor. Table 1-2 lists allowable stress values for a few materials at temperatures below the creep and rupture values. Allowable stresses at elevated temperatures where creep and rupture are controlling factors are discussed in Chapter 2. Allowable stress values for reinforced concrete are given in various standards such as ACI 318.

Maximum moments and stresses in most of the frequently encountered load cases for simply supported rectangular plates have been tabulated in many references for the convenience of the engineer. Timoshenko (Timoshenko and Woinowsky-Krieger 1959) lists tables and charts for maximum moments and deflections of numerous loading conditions. Roark (Roark and Young 1975) has similar tables and so does Pilkey (Pilkey and Chang 1978). Table 1-3 gives maximum deflection and stress values for simply supported plates with two commonly encountered loading conditions.

Loading conditions not found in published references must be solved by developing a Fourier series for the loads and deflection and then satisfying the boundary conditions as discussed in this chapter.

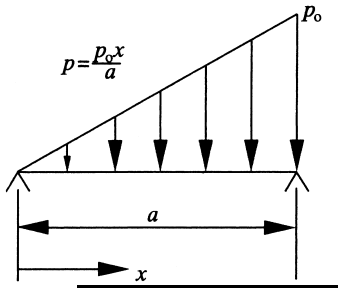
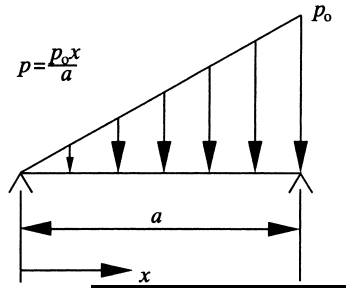
The exact analysis of perforated rectangular plates, which are used in boilers and pressure vessels, is difficult to obtain. However, various approximations can

Table 1-2. ASME VIII–1 allowable stress values (ksi)

Material	ASME* Designation	Temperature, °F			
		Room Temperature	300	500	700
Carbon steel	SA 516–70	20.0	20.0	20.0	18.1
Stainless steel	SA 240–304	20.0	18.9	17.5	15.8
Aluminum SB	209–6061 T4	7.8	6.2		
Copper Alloy	SB 171–715 70/30	13.3	12.0	11.0	10.4
Nickel Alloy	SB 575–276	27.3	27.3	26.9	24.0
Titanium alloy	SB 265–Gr3	18.6	12.8	9.3	

*ASME uses SA and SB designations for ferrous and non-ferrous alloys to distinguish these specifications from ASTM's A and B specifications. The alloys shown in this table, however, have identical properties in the SA, SB, and A, B specifications.

Table 1-3. Simply supported rectangular plates.

		Uniform pressure, p_o , over total surface					
a/b	K_1	K_2	K_1	K_2	b/a	K_1	K_2
1.0	0.2873	0.04436	0.1619	0.02243	1.0	0.1619	0.02243
1.1	0.3291	0.05317	0.1710	0.02697	1.1	0.1838	0.02682
1.2	0.3761	0.06170	0.1938	0.03142	1.2	0.2045	0.03108
1.3	0.4163	0.06980	0.2160	0.03570	1.3	0.2239	0.03511
1.4	0.4533	0.07737	0.2376	0.03978	1.4	0.2417	0.03888
1.5	0.4872	0.08435	0.2568	0.04362	1.5	0.2578	0.04215
1.6	0.5174	0.09073	0.2742	0.04723	1.6	0.2725	0.04555
1.8	0.5691	0.1017	0.3072	0.05375	1.8	0.2976	0.05103
2.0	0.6101	0.1106	0.3358	0.05943	2.0	0.3174	0.05546
2.5	0.6776	0.1255	0.3922	0.07077	2.5	0.3498	0.06291
3.0	0.7134	0.1336	0.4341	0.07920	3.0	0.3670	0.06693
3.5	0.7314	0.1378	0.4664	0.08570	3.5	0.3759	0.06904
4.0	0.7410	0.1400	0.4922	0.09089	4.0	0.3804	0.07012
6.0	0.7493	0.1421	0.5588	0.1042	6.0	0.3847	0.07116

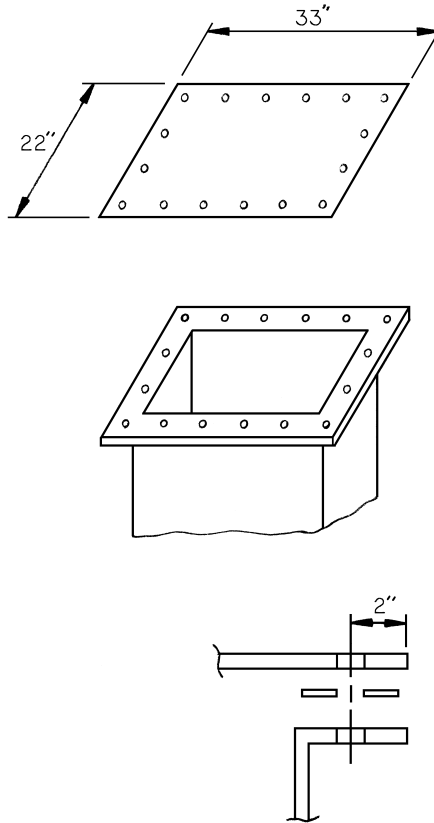
Notation: Maximum stress $S = K_1 p_o b^2 / t^2$; maximum deflection $w = K_2 p_o b^4 / Et^3$. a = plate dimension; b = plate dimension; E = modulus of elasticity; K_1 = stress factor ($\mu = 0.3$); K_2 = deflection factor ($\mu = 0.3$); p_o = maximum pressure; S = maximum stress; t = thickness; w = maximum deflection.

be made to obtain a solution. One such approximation is given by the ASME boiler code, Section I, and consists of using the classical solution of a solid plate and then modifying it by using ligament efficiency factors to account for the effect of the perforations on the plate stress.

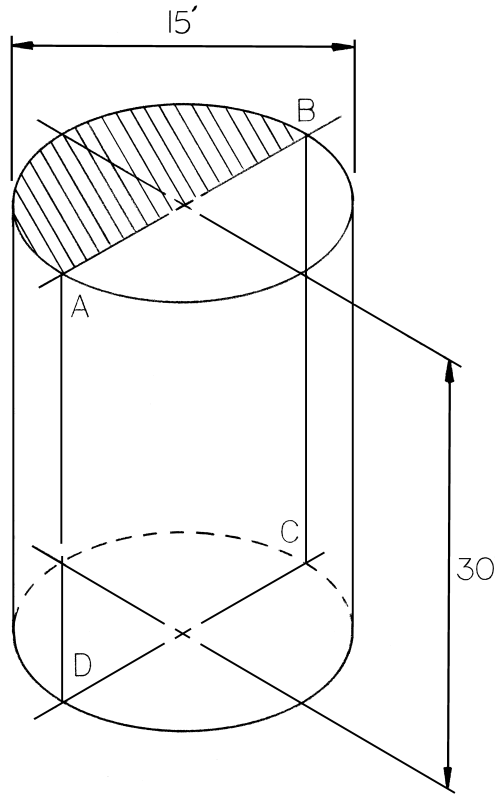
Problems

1-14 What is the required thickness of the rectangular cover for the opening shown? The cover is made of aluminum SA209-6061 T4 material. The temperature is 300°F and the applied external pressure is 10 psi. Calculate the required number of bolts if they are made of the same material. Do the corner bolts have to be larger than the rest of the bolts?

1-15 Find the required thickness of the internal partition $ABCD$ of the holding tank. The bottom of the tank is enclosed by a circular plate while the top is



Prob. 1-14. Rectangular cover.



Prob. 1-15. Internal partition in a cylindrical shell.

enclosed by a semicircular plate that covers one side of the top. The tank is full of water on one side of the partition. The material is SA 240-304 and the temperature is 100°F . Assume the partition $ABCD$ to be simply supported on all four sides.

1-16 In Example 1-2, what modifications must be made to Eq. (1) if the thickness of the plate is variable and is a function of x and y .

1-17 Discuss the modifications that have to be made to the differential equation and boundary conditions if the corners of a simply supported plate are allowed to curl up due to applied pressure.

1-18 In Fig. 1-14, how should the reinforcing bars in a concrete slab be placed to ensure continuity of moments, shears, etc. between the slab and beam?

Bending of Various Rectangular Plates

2-1 Plates with Various Boundary Conditions

The majority of rectangular plates in building slabs, ship hulls, aircraft skins (Fig. 2-1), and rectangular holding tanks have boundary conditions other than simply supported. The Levy solution discussed in Chapter 1 can be utilized very effectively in solving rectangular plates with various boundary conditions. Equations (1-46) and (1-50) are readily applicable to plates with two opposite sides simply supported, and the boundary conditions for the other two sides can then be incorporated into the total solution. For plates that do not have two opposite simply supported sides, the solution is more difficult because various cases have to be superimposed to arrive at a solution. The following examples illustrate the general procedure to be followed in solving plates with various boundary conditions.

Example 2-1

Find the expression for the deflection of a uniformly loaded plate having three sides simply supported and the fourth side fixed as shown in Fig. 2-2.

Solution

From Example 1-4, the particular deflection is expressed as

$$w_p = \sum_{m=1,3,\dots}^{\infty} \frac{4p_o a^4}{m^5 \pi^5 D} \sin \frac{m\pi x}{a}$$



Figure 2-1. F-15 Fighter. (Courtesy of the Boeing Corp.)

and the total deflection is given by

$$w = \sum_{m=1,2,\dots}^{\infty} \left(A_m \sinh \frac{m\pi y}{a} + B_m \cosh \frac{m\pi y}{a} + C_m y \sinh \frac{m\pi y}{a} + D_m y \cosh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a} + \sum_{m=1,3,\dots}^{\infty} \frac{4p_o a^4}{m^5 \pi^5 D} \sin \frac{m\pi x}{a}.$$

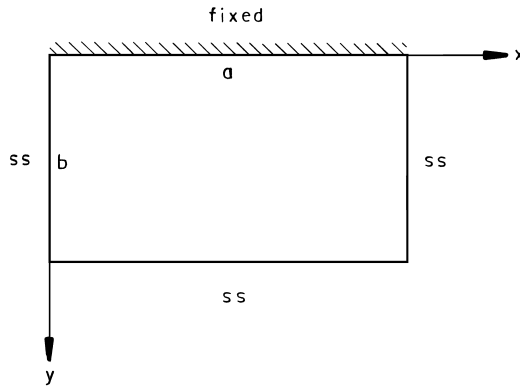


Figure 2-2. Rectangular plate with various boundary conditions.

Since the deflection is symmetric with respect to $x = a/2$, m must be odd and the deflection expression becomes

$$w = \sum_{m=1,3,\dots}^{\infty} \left(A_m \sinh \frac{m\pi y}{a} + B_m \cosh \frac{m\pi y}{a} + C_m y \sinh \frac{m\pi y}{a} + D_m y \cosh \frac{m\pi y}{a} + \frac{4p_o a^4}{m^5 \pi^5 D} \right) \sin \frac{m\pi x}{a}. \quad (1)$$

The boundary conditions are given by

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial y} = 0 \quad \text{at} \quad y = 0$$

$$w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at} \quad y = b.$$

Using these four boundary conditions to solve for the unknown constants in Eq. (1) gives

$$A_m = \frac{4p_o a^4}{m^5 \pi^5 D} \frac{F_m}{G_m}$$

$$B_m = \frac{-4p_o a^4}{m^5 \pi^5 D}$$

$$C_m = \frac{4p_o a^4}{m^5 \pi^5 D} \frac{H_m}{G_m}$$

$$D_m = -\frac{m\pi}{a} A_m$$

where,

$$F_m = 2 \cosh^2 \left(\frac{m\pi b}{a} \right) - 2 \cosh \frac{m\pi b}{a} - \frac{m\pi b}{a} \sinh \frac{m\pi b}{a}$$

$$G_m = 2 \cosh \frac{m\pi b}{a} \sinh \frac{m\pi b}{a} - 2 \frac{m\pi b}{a}$$

$$H_m = 2 \frac{m\pi}{a} \sinh \frac{m\pi b}{a} \cosh \frac{m\pi b}{a} - \frac{m\pi}{a} \sinh \frac{m\pi b}{a} - \left(\frac{m\pi}{a} \right)^2 b \cosh \frac{m\pi b}{a}.$$

With these constants established, Eq. (1) can be solved for the deflection. The moments throughout the plate can then be obtained from Eq. (1-17).

Example 2-2

For the plate shown in Fig. 2-3, find the maximum bending moment if $p = 10$ psi, $a = 30$ inches, $b = 24$ inches, $E = 30,000$ ksi, and $\mu = 0.3$. Use two terms of the series only. What is the required thickness if the allowable stress is 20,000 psi?

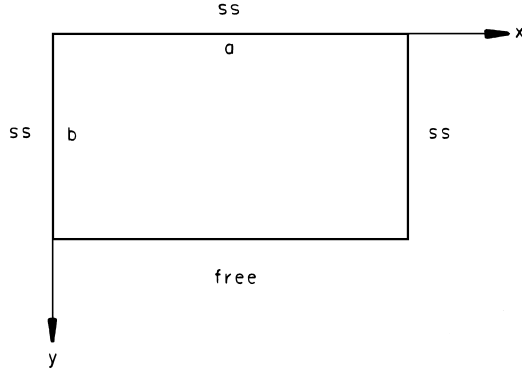


Figure 2-3. Plate with one free and three simply supported edges.

Solution

The boundary conditions are

$$(1) \text{ at } y = 0, \quad w = 0$$

$$(2) \text{ at } y = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0$$

$$(3) \text{ at } y = b, \quad \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2} = 0$$

$$(4) \text{ at } y = b, \quad \frac{\partial^3 w}{\partial y^3} + (2 - \mu) \frac{\partial^3 w}{\partial x^2 \partial y} = 0.$$

Since the solution is symmetric around $x = a/2$, m must be odd and the general solution is given by

$$w = \sum_{m=1,3,\dots}^{\infty} \left(A_m \sinh \frac{m\pi y}{a} + B_m \cosh \frac{m\pi y}{a} + C_m y \sinh \frac{m\pi y}{a} + D_m y \cosh \frac{m\pi y}{a} + \frac{4p_o a^4}{m^5 \pi^5 D} \right) \sin \frac{m\pi x}{a}. \quad (1)$$

From the first boundary condition, Eq. (1) gives

$$B_m = \frac{-4p_o a^4}{m^5 \pi^5 D}.$$

From the second boundary condition, Eq. (1) gives

$$C_m = \frac{2p_o a^3}{m^4 \pi^4 D}.$$

Solving the third and fourth boundary conditions yields

$$A_m = \frac{K_m G_m - J_m H_m}{I_m G_m - J_m F_m}$$

and

$$D_m = \frac{H_m - A_m F_m}{G_m}$$

where

$$F_m = (1 - \mu) \frac{m^2 \pi^2}{a^2} \sinh \frac{m\pi b}{a}$$

$$G_m = \frac{2m\pi}{a} \sinh \frac{m\pi b}{a} + \frac{m^2 \pi^2}{a^2} b (1 - \mu) \cosh \frac{m\pi b}{a}$$

$$H_m = B_m \frac{m^2 \pi^2}{a^2} (\mu - 1) \cosh \frac{m\pi b}{a} + C_m \frac{m^2 \pi^2}{a^2} b (\mu - 1) \sinh \frac{m\pi b}{a} \\ - \frac{2m\pi}{a} C_m \cosh \frac{m\pi b}{a} + \frac{4\pi p_o a^2}{m^3 \pi^3 D}$$

$$I_m = (\mu - 1) \frac{m^3 \pi^3}{a^3} \cosh \frac{m\pi b}{a}$$

$$J_m = (1 + \mu) \frac{m^2 \pi^2}{a^2} \cosh \frac{m\pi b}{a} + (\mu - 1) b \frac{m^3 \pi^3}{a^3} \sinh \frac{m\pi b}{a}$$

$$K_m = (1 - \mu) \frac{m^3 \pi^3}{a^3} B_m \sinh \frac{m\pi b}{a} - (1 + \mu) \frac{m^2 \pi^2}{a^2} C_m \sinh \frac{m\pi b}{a} \\ + b \frac{m^3 \pi^3}{a^3} C_m (1 - \mu) \cosh \frac{m\pi b}{a}.$$

The bending moment is obtained from Eq. (1) and the expression

$$M_x = -D \left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right).$$

It has a maximum value at $y = b$ and $x = a/2$. Hence,

$$\frac{M_x}{D} = \sum_{m=1,3,\dots}^{\infty} \left\{ \left(A_m \sinh \frac{m\pi b}{a} + B_m \cosh \frac{m\pi b}{a} + C_m b \sinh \frac{m\pi b}{a} \right. \right. \\ \left. \left. + D_m b \cosh \frac{m\pi b}{a} + \frac{4p_o a^4}{m^5 \pi^5 D} \right) \frac{m^2 \pi^2}{a^2} \right. \\ \left. - \mu \left[\frac{m\pi}{a} \left(\frac{m\pi}{a} A_m + 2D_m \right) \sinh \frac{m\pi b}{a} \right. \right. \\ \left. \left. + \frac{m\pi}{a} \left(B_m \frac{m\pi}{a} + 2C_m \right) \cosh \frac{m\pi b}{a} + C_m b \frac{m^2 \pi^2}{a^2} \sinh \frac{m\pi b}{a} \right. \right. \\ \left. \left. + D_m b \frac{m^2 \pi^2}{a^2} \cosh \frac{m\pi b}{a} \right] \sin \frac{m\pi}{2} \right\} \sin \frac{m\pi}{2}. \quad (2)$$

The stiffness factor D can be deleted from the lefthand side of Eq. (2) and from constants A_m , B_m , C_m , and D_m on the righthand side of the equation. Equation (2)

is solved by calculating all constants for $m = 1$ and $m = 3$. This can best be done in tabular form as

Value	$m = 1$	$m = 3$
DB_m	-105,875.54	-435.7
DC_m	5543.63	68.44
F_m	0.04707	64.99
G_m	2.429	2150.94
DH_m	-8078.68	-118,879.48
I_m	-4.995E-3	-20.4184
J_m	-0.02973	-369.34
DK_m	-342.02	16,381.24
DA_m	99,789.12	435.47
DD_m	-5259.82	-68.43
M_x	913	-46

Total $M_x = 867$ inch-lbs/inch

$$t = \sqrt{6M/\sigma} = \sqrt{6 \times 867/20,000}$$

$$= 0.51 \text{ inch}$$

Example 2-3

Find the expression for the deflection of the plate shown in Fig. 2-4a due to a uniform load.

Solution

The solution must be divided into three separate cases as shown in Fig. 2-4b. The differential equations to be solved are

$$\nabla^4 w_1 = p/D, \quad \nabla^4 w_2 = 0, \quad \nabla^4 w_3 = 0$$

and

$$w = w_1 + w_2 + w_3. \quad (1)$$

Case 1

The solution of a uniformly loaded, simply supported plate is given in Example 1-4. Using the coordinate system shown in Fig. 2-4b, the solution is expressed as

$$w_1 = \sum_{m=1,3,\dots}^{\infty} \left(B_m \cosh \frac{m\pi(y-b/2)}{a} + C_m (y-b/2) \sinh \frac{m\pi(y-b/2)}{a} \right. \\ \left. + \frac{4p_o a^4}{m^5 \pi^5 D} \right) \sin \frac{m\pi x}{a} \quad (2)$$

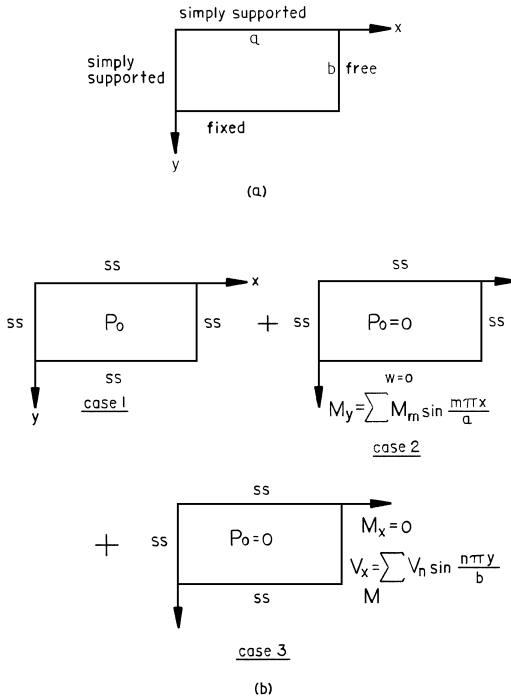


Figure 2-4. Plate with various boundary conditions.

or

$$w_1 = \sum_{m=1,3,\dots}^{\infty} J_m(y) \sin \frac{m\pi x}{a} \quad (3)$$

where

$$J_m(y) = \sum_{n=1,3,\dots}^{\infty} \left(B_m \cosh \frac{m\pi(y-b/2)}{a} + C_m (y-b/2) \sinh \frac{m\pi(y-b/2)}{a} + \frac{4p_o a^4}{m^5 \pi^5 D} \right)$$

$$C_m = \frac{2a^3 p_o}{m^4 \pi^4 D \cosh \frac{m\pi b}{2a}}$$

$$B_m = \frac{4a^4 p_o + m\pi p_o a^3 b \tanh \frac{m\pi b}{2a}}{m^5 \pi^5 D \cosh \frac{m\pi b}{2a}}.$$

Case 2

The boundary conditions are

$$\frac{\partial^2 w_2}{\partial y^2} \Big|_{y=0} = w_2|_{y=0} = w_2|_{y=b} = 0$$

and

$$-D \left(\frac{\partial^2 w_2}{\partial y^2} + \mu \frac{\partial^2 w_2}{\partial x^2} \right) \bigg|_{y=b} = \sum_{m=1,2,\dots}^{\infty} M_m \sin \frac{m\pi x}{a}.$$

Substituting the function

$$w_2 = \sum_{m=1,2,\dots}^{\infty} f_m(y) \sin \frac{m\pi x}{a}$$

into the differential equation $\nabla^4 w_2 = 0$ gives

$$f_m(y) = A_m \sinh \frac{m\pi y}{a} + B_m \cosh \frac{m\pi y}{a} + C_m y \sinh \frac{m\pi y}{a} + D_m y \cosh \frac{m\pi y}{a}.$$

Substituting the deflection expression into the first three boundary conditions gives

$$\begin{aligned} B_m &= C_m = 0 \\ A_m &= -D_m b \coth \frac{m\pi b}{a}, \end{aligned}$$

and from the fourth boundary condition we get

$$D_m = -\frac{M_m}{DT_m}$$

where

$$T_m = \frac{2m\pi}{a} \sinh \frac{m\pi b}{a}.$$

Hence,

$$w_2 = \sum_{m=1,2,\dots}^{\infty} \frac{M_m}{DT_m} \left(b \coth \frac{m\pi b}{a} \sinh \frac{m\pi y}{a} - y \cosh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a}, \quad (4)$$

which can be written as

$$w_2 = \sum_{m=1,2,\dots}^{\infty} K_m(y) \sin \frac{m\pi x}{a}. \quad (5)$$

Case 3

The boundary conditions are

$$w_3 = \frac{\partial^2 w_3}{\partial x^2} \bigg|_{x=0} = \left(\frac{\partial^2 w_3}{\partial x^2} + \frac{\partial^2 w_3}{\partial y^2} \right) \bigg|_{x=a} = 0$$

and

$$\left(\frac{\partial^3 w_3}{\partial x^3} + (2 - \mu) \frac{\partial^3 w_3}{\partial x \partial y^2} \right) \bigg|_{x=a} = \sum_{n=1,2,\dots}^{\infty} V_n \sin \frac{n\pi y}{b}.$$

Substituting the function

$$w_3 = \sum_{n=1,2,\dots}^{\infty} g_n(x) \sin \frac{n\pi y}{b}$$

into the differential equation $\nabla^4 w_3 = 0$ gives

$$g_n(x) = A_n \sinh \frac{n\pi x}{b} + B_n \cosh \frac{n\pi x}{b} + C_n x \sinh \frac{n\pi x}{b} + D_n x \cosh \frac{n\pi x}{b}.$$

Substituting this expression into the four boundary conditions results in an expression similar to that of Eq. (5) and can be expressed as

$$w_3 = \sum_{n=1,2,\dots}^{\infty} L_n(x) \sin \frac{n\pi y}{b}. \quad (6)$$

Substituting Eqs. (3), (5), and (6) into Eq. (1) gives

$$w = \sum_{m=1,3,\dots}^{\infty} J_m(y) \sin \frac{m\pi x}{a} + \sum_{m=1,2,\dots}^{\infty} K_m(y) \sin \frac{m\pi x}{a} + \sum_{n=1,2,\dots}^{\infty} L_n(x) \sin \frac{n\pi y}{b}. \quad (7)$$

This equation has two unknowns. They are M_m in the expression $K_m(y)$, and V_n in the expression $L_n(x)$. The boundary conditions of Fig. 2-4a are

$$\text{at } x = 0, w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial x^2} = 0$$

$$\text{at } x = a, M_x = 0 \quad \text{and} \quad V_x = 0$$

$$\text{at } y = 0, w = 0 \quad \text{and} \quad \frac{\partial^2 w}{\partial y^2} = 0$$

$$\text{at } y = b, w = 0 \quad \text{and} \quad \frac{\partial w}{\partial y} = 0.$$

Equation (7) satisfies all of the above boundary conditions except

$$\left. \frac{\partial w}{\partial y} \right|_{y=b} = 0 \quad \text{and} \quad V_x|_{x=a} = 0.$$

These two boundary conditions are used to determine the two unknowns in Eq. (7). From the first boundary condition we get

$$\left(\sum_{m=1,3,\dots}^{\infty} \frac{dJ_m}{dy} \sin \frac{m\pi x}{a} + \sum_{m=1,2,\dots}^{\infty} \frac{dK_m}{dy} \sin \frac{m\pi x}{a} + \sum_{n=1,2,\dots}^{\infty} \frac{n\pi}{b} L_n \cos \frac{n\pi y}{b} \right) \bigg|_{y=b} = 0. \quad (8)$$

In order to solve this expression, the last term needs to be expressed in terms of the quantity $\sin(\frac{m\pi x}{a})$. This can be accomplished by letting

$$L_n(x) = \sum_{m=1,2,\dots}^{\infty} H_{mn} \sin \frac{m\pi x}{a}$$

where

$$H_{mn} = \frac{2}{a} \int_0^a L_n(x) \sin \frac{m\pi x}{a} dx.$$

Defining

$$\cos \frac{n\pi y}{b} = (-1)^n \text{ at } y = b$$

and

$$\begin{aligned} \delta_m &= 0 & \text{when } m \text{ is even,} \\ \delta_m &= 1 & \text{when } m \text{ is odd,} \end{aligned}$$

Eq. (8) becomes

$$\sum_{m=1,2,\dots}^{\infty} \left[\delta_m \frac{dJ_m}{dy} \right]_{y=b} + \frac{dK_m}{dy} \bigg|_{y=b} + \sum_{n=1,2,\dots}^{\infty} \frac{n\pi}{b} H_{mn} (-1)^n \left[\sin \frac{m\pi x}{a} = 0 \right]$$

or

$$\delta_m \frac{dJ_m}{dy} \bigg|_{y=b} + \frac{dK_m}{dy} \bigg|_{y=b} + \sum_{n=1,2,\dots}^{\infty} \frac{n\pi}{b} H_{mn} (-1)^n = 0. \quad (9)$$

This equation cannot be solved directly for V_n and M_m . Rather it has infinite solutions of M_m and V_n . Thus, if we truncate the equation after $m = n = 2$, then the equation becomes

$$\frac{dJ_1}{dy} \bigg|_{y=b} + \frac{dK_1}{dy} \bigg|_{y=b} - \frac{\pi}{b} H_{11} + \frac{2\pi}{b} H_{12} = 0$$

and

$$\frac{dJ_2}{dy} \bigg|_{y=b} + \frac{dK_2}{dy} \bigg|_{y=b} - \frac{\pi}{b} H_{21} + \frac{2\pi}{b} H_{22} = 0.$$

Similarly two other equations can be written to satisfy the boundary condition

$$V_x = 0.$$

From the four simultaneous equations, the unknown quantities M_1 , M_2 , V_1 , and V_2 are determined from the expressions for K_m and L_m . Once these expressions are known, the deflection is obtained from Eq. (7).

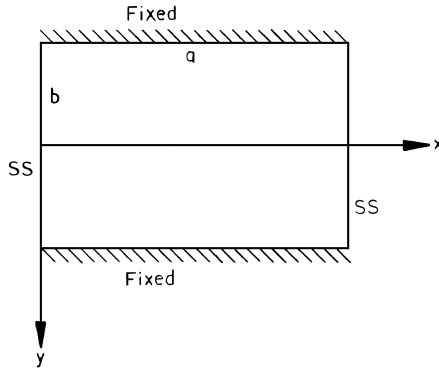
Problems

2-1 Use the expressions derived in Example 2-1 to determine the maximum stress in a rectangular plate with $a = 90$ cm, $b = 60$ cm, $p = 1$ kgf/cm², $E = 21,000$ kgf/mm², and $\mu = 0.3$. Let $t = 1.5$ cm.

2-2 Use the expressions derived in Example 2-2 to determine the maximum stress in a titanium rectangular plate with $a = 40$ inches, $b = 30$ inches, $p = 30$ psi, $E = 15,000$ ksi, and $\mu = 0.32$. Let $t = 1.0$ inch.

2-3 Use the expressions derived in Example 2-2 to determine the maximum stress at a point where $x = 15$ inches and $y = 15$ inches. Let $a = 40$ inches, $b = 50$ inches, $E = 30,000$ ksi, $p = 10$ psi, $t = 1.0$ inch, and $\mu = 0.30$.

2-4 A copper internal baffle plate in a reactor has the boundary conditions shown in Fig. 2-4. Find its maximum deflection if $a = 36$ inches, $b = 20$ inches, $p = 100$ psi, $E = 16,000$ ksi, and $\mu = 0.33$. Let $t = 2.0$ inches.



Prob. 2-5. Plate with two edges fixed and the other two simply supported.

2-5 The top plate of a truck weigh scale is supported by beams such that the edge conditions can be approximated as shown. Find the expressions for the bending moments at the edges and in the middle due to uniform pressure p .

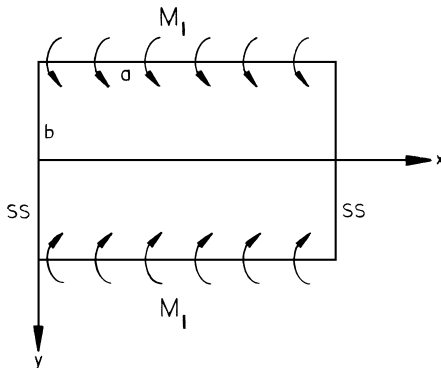
2-6 Find the expression for the deflection of a rectangular plate due to edge moments given by

$$M_1 = \sum_{m=1}^{\infty} E_m(y) \sin \frac{m\pi x}{a}.$$

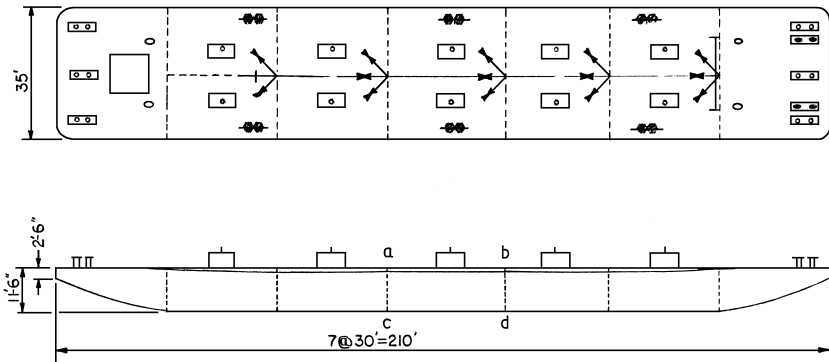
2-7 Find the expression for the deflection of a rectangular plate fixed at all sides.

Hint: Use the results of Problem 2-6 to solve this problem by switching the x - and y -axes.

2-8 The bulk concrete barge shown consists of various compartments. Assume all compartments are half-filled to height 5' 9". Assume the specific gravity of the



Prob. 2-6. Plate with applied moments on two opposite edges.



Prob. 2-8. Oil barge.

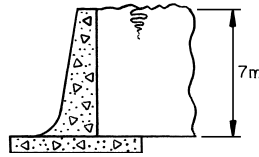
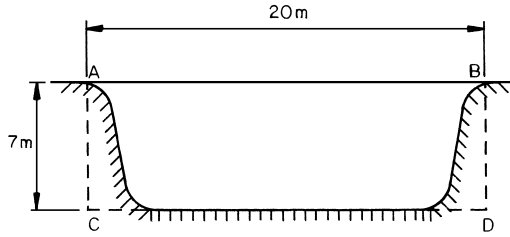
contents to be twice that of the water outside. The equivalent pressure distribution on the side panels is triangular in shape with a value of zero at the top and bottom lines, and increases linearly to a maximum value of 2.50 psi halfway down the side panel ($0.0361 \text{ lbs per cubic inch} \times 5.75 \text{ ft} \times 12 \text{ in/ft}$). Find the moment in the side plate $abcd$ due to this triangular distribution. Assume the panel to be simply supported at the top and bottom lengths and fixed along the sides. Also, assume the length of the panel to be 360 inches which is the spacing of the bulkheads in the barge (this large length disregards the effects of the intermediate vertical stiffeners welded to the side plates to increase rigidity inbetween the bulkheads as shown in Problem 4-10). Let $E = 29,000 \text{ ksi}$ and $\mu = 0.29$.

2-9 Find the maximum moment in the small concrete dam due to hydrostatic pressure. Sides AC and BD are simply supported. Side AB is free and side CD is fixed. Let $E = 2180 \text{ kgf/mm}^2$ and $\mu = 0$. Assume a uniform thickness.

2-2 Continuous Plates

The classical methods developed so far for solving rectangular plates are also applicable to continuous plates. The boundary conditions of each panel as well as the compatibility of forces or deformations between any two panels across a common boundary, such as ab in Fig. 2-5a and ab and ac in Fig. 2-5b, must be used to determine the constants in the differential equations of each panel. As the number of panels increases, it becomes more tedious to find a solution with the classical plate theory due to the number of simultaneous differential equations that must be solved to obtain the constants of integration. A more practical approach for solving such plates is to use an approximate solution such as the finite difference method discussed in Chapter 5 or a finite element analysis which is discussed in Chapter 16.

The procedure for solving continuous plates is illustrated in Example 2-4 for a two-panel structure.



Prob. 2-9. Concrete dam.

Example 2-4

Find the expressions for the deflection in the continuous plate shown in Fig. 2-6a due to a uniform load on panel I only.

Solution

The boundary conditions for panel I are

$$\text{at } y_1 = 0 \quad w = 0 \quad (1)$$

$$\frac{\partial^2 w}{\partial y_1^2} = M_o \quad (2)$$

$$\text{at } y_1 = b \quad w = 0 \quad (3)$$

$$\frac{\partial w}{\partial y_1} = 0. \quad (4)$$

The expression for the deflection is obtained from Example 2-1 as

$$w = \sum_{m=1,3,\dots} \left(A_m \sinh \frac{m\pi y}{a} + B_m \cosh \frac{m\pi y}{a} + C_m y \sinh \frac{m\pi y}{a} + D_m y \cosh \frac{m\pi y}{a} + \frac{4p_o a^4}{m^5 \pi^5 D} \right) \sin \frac{m\pi x}{a}. \quad (5)$$

Let the moment M_o between panels I and II be represented by

$$M_o = \sum_{m=1,3,\dots}^{\infty} E_m(y) \sin \frac{m\pi x}{a}. \quad (6)$$

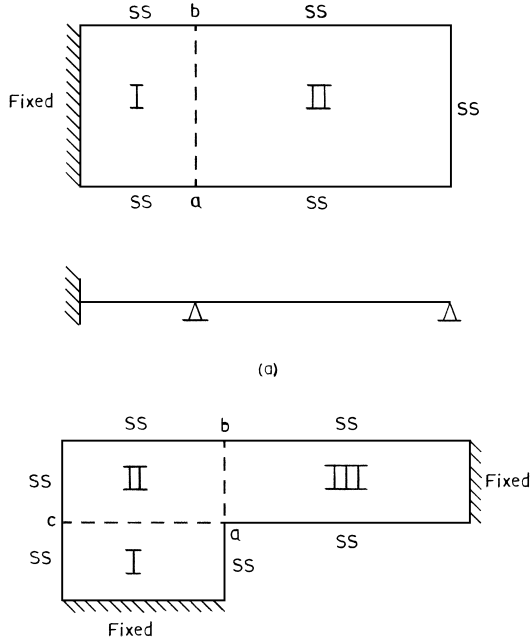


Figure 2-5. Continuous plate.

From the boundary conditions (1) through (4) we get

$$A_m = -B_m \left(\cosh \frac{m\pi b}{a} - 1 \right) - C_m b \sinh \frac{m\pi b}{a} - D_m b \cosh \frac{m\pi b}{a} \quad (7)$$

$$B_m = -\frac{4p_o a^4}{m^5 \pi^5 D} \quad (8)$$

$$C_m = \frac{-E_m}{2 \frac{m\pi}{a}} + \frac{2p_o a^3}{m^4 \pi^4 D} \quad (9)$$

$$D_m = \frac{K_1}{K_2} B_m + C_m \frac{\sinh^2 \frac{m\pi b}{a}}{K_2} \quad (10)$$

where

$$K_1 = \frac{m\pi}{a} \left(\cosh \frac{m\pi b}{a} - 1 \right)$$

$$K_2 = \frac{m\pi b}{a} - \sinh \frac{m\pi b}{a} \cosh \frac{m\pi b}{a}.$$

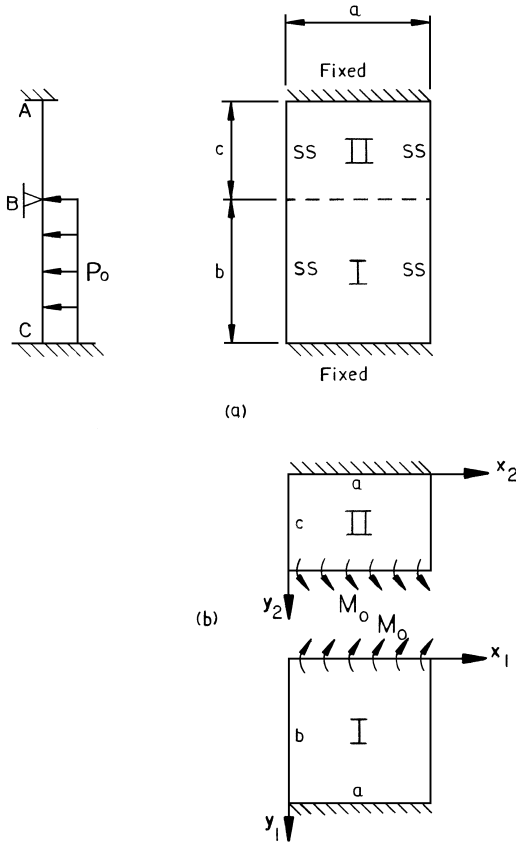


Figure 2-6. Continuous plate with partial load.

The boundary conditions for panel II are

$$\text{at } y_2 = 0 \quad w = 0 \quad (11)$$

$$\frac{\partial w}{\partial y_2} = 0 \quad (12)$$

$$\text{at } y_2 = c \quad w = 0 \quad (13)$$

$$\frac{\partial^2 w}{\partial y_2^2} = M_0 \quad (14)$$

The expression for the deflection is written as

$$w = \sum_{m=1,3,\dots}^{\infty} \left(F_m \sinh \frac{m\pi y}{a} + G_m \cosh \frac{m\pi y}{a} + H_m y \sinh \frac{m\pi y}{a} + I_m y \cosh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a}. \quad (15)$$

From the boundary conditions (11) through (14) we get

$$F_m = -\frac{E_m}{K_3 + K_4 K_5} \quad (16)$$

$$G_m = 0 \quad (17)$$

$$H_m = \frac{1}{c} \left(1 - \frac{m\pi c}{a} \coth \frac{m\pi c}{a} \right) \frac{E_m}{K_3 + K_4 K_5} \quad (18)$$

$$I_m = \frac{m\pi}{a} \frac{E_m}{K_3 + K_4 K_5} \quad (19)$$

where

$$K_3 = \frac{m^2 \pi^2}{a^2} \left(\sinh \frac{m\pi c}{a} + \frac{m\pi c}{a} \cosh \frac{m\pi c}{a} \right)$$

$$K_4 = \frac{\sinh \frac{m\pi c}{a} - \frac{m\pi c}{a} \cosh \frac{m\pi c}{a}}{c \sinh \frac{m\pi c}{a}}$$

$$K_5 = \frac{2m\pi}{a} \cosh \frac{m\pi c}{a} + \frac{m^2 \pi^2}{a^2} c \sinh \frac{m\pi c}{a}.$$

The constants A_m through I_m are in terms of the bending moment constant E_m . This constant is obtained by solving the compatibility equation at the common boundary which is

$$\text{slope in panel I} = \text{slope in panel II.}$$

Once E_m is known, then equations (5) and (15) can be solved for the deflections at any location in the plates.

Other applications of the continuous plate concept are for large plates with multiple point supports. Such applications are found in concrete flat slab floors (Winter et al. 1964) with multiple column supports as well as stayed vessels commonly encountered in chemical plants and refineries. These cylindrical vessels consist of inner and outer shells tied together with stays and the annular space between them pressurized. The analysis of both inner and outer shells is based on the theory of plates with multiple-point supports.

Numerous articles have been written on the subject of plates with multiple supports. Some of these articles are listed in Timoshenko (Timoshenko and Woinowsky-Krieger 1959). A large plate under uniform pressure with closely spaced supports (Fig. 2-7a) can be analyzed using the Levy solution. Due to symmetry, the boundary conditions in the plate along $x = \pm a/2$ are

$$\text{slope} = \frac{\partial w}{\partial x} = 0 \quad (2-1)$$

and from Eq. (1-27),

$$\text{shear } Q_x = -D \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) = 0. \quad (2-2)$$

Similar boundary conditions exist along the boundary $y = \pm a/2$. The particular solution of the equation

$$\nabla^4 w = q/D \quad (2-3)$$

and the homogeneous solution of

$$\nabla^4 w = 0 \quad (2-4)$$

are accomplished by expressing the deflection by the series

$$w = f_0 + \sum_{m=2,4,\dots}^{\infty} f_m(y) \cos \frac{m\pi x}{a} \quad (2-5)$$

and the load by

$$p = p_0 + \sum_{m=2,4,\dots}^{\infty} p_m(y) \cos \frac{m\pi x}{a}. \quad (2-6)$$

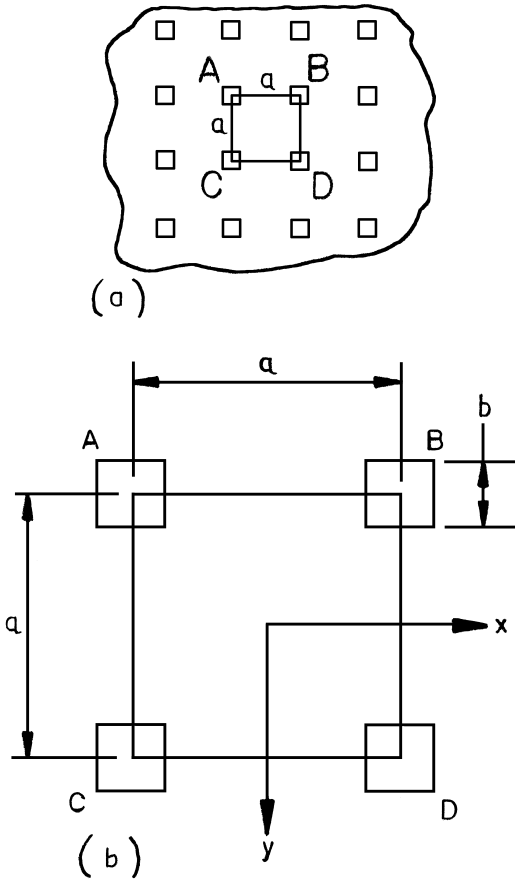


Figure 2-7. Plate on multiple supports.

Equation (2-5) satisfies the boundary conditions given by Eqs. (2-1) and (2-2). Substituting Eqs. (2-5) and (2-6) into Eqs. (2-3) and (2-4) and summing the two equations results in an expression for the deflection with three unknown constants. The first is obtained directly from the slope boundary in Eq. (2-1). The second constant is obtained from Eq. (2-2) keeping in mind that the shear, Q , is equal to zero along the unsupported length $(a-b)$ and $pa^2/4$ over the supports b . The third unknown constant is evaluated from the condition that the deflection, w , is equal to zero at the supports.

With the deflection known, the bending moments can be determined from Eq. (1-17). The maximum negative bending moment occurs at the supports and is expressed by Woinowsky-Krieger as

$$M_{\max} = (1 + \mu)pa^2 \left[\frac{(1-r)(2-r)}{48} + \frac{1}{4\pi^3 r^2} \sum_{m=1}^{\infty} \frac{2}{m^3 \sinh m\pi} \sinh \frac{m\pi r}{2} \cosh \frac{m\pi(2-r)}{2} \sin m\pi r \right] \quad (2-7)$$

where $r = b/a$. Figure 2-8 shows a plot of this equation for various r ratios.

The analysis expressed by Eq. (2-7) assumes the intermediate supports to be rigid. If the supports deflect due to applied loads, then the moments must be determined from an analysis of a plate on an elastic foundation.

Problems

2-10 Find the expressions for the bending moments in Example 2-4 if $a = 5$ m, $b = 15$ m, $c = 10$ m, $p = 1$ kgf/cm², $E = 210,000$ MPa, and $\mu = 0$.

2-11 Plot the value of M_x along the length of panels I and II in Fig. 2-6a at $x = a/2$. Compare the result to the bending moments obtained from a beam of unit width and length ABC in Fig. 2-6a.

2-12 A continuous concrete slab is supported by columns as shown in Fig. 2-7. Calculate the maximum bending moment if $a = 20$ ft, $b = 3$ ft, $p = 150$ psf, and $\mu = 0$.

2-3 Plates on an Elastic Foundation

The effective pressure on any point in a plate or slab resting on a continuous foundation such as a concrete road pavement or a rectangular tubesheet in a heat exchanger (Fig. 2-9), is equal to $p - f$ where p is the applied load and f is the resisting pressure of the foundation. If we assume the foundation to be elastic, i.e., its elasticity is defined by a force that causes a unit deflection when applied over a unit area, then we can define

$$f = kw \quad (2-8)$$

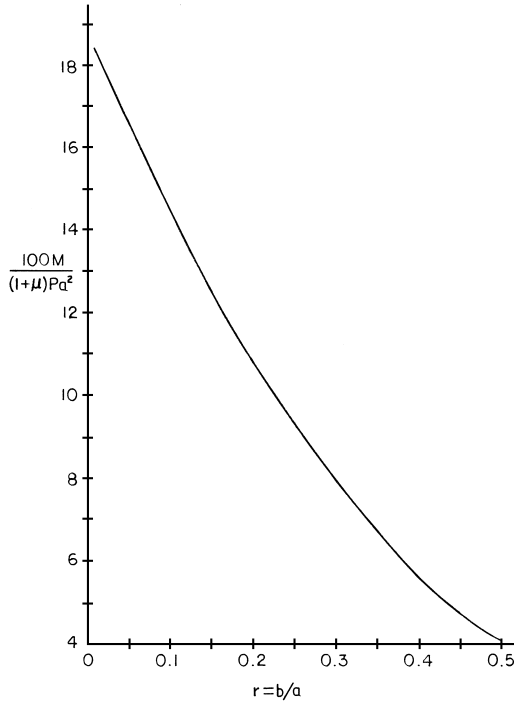


Figure 2-8. A plot of Eq.(2-7).

where k is the foundation modulus. Values of k for various soils are given in numerous references such as McFarland (McFarland et al. 1972).

Equation (1-26) for the lateral deflection of a plate can now be modified to

$$\nabla^4 w = \frac{1}{D}(p - kw)$$

or

$$\nabla^4 w + \frac{k}{D}w = p/D \quad (2-9)$$

It must be kept in mind that under certain applied loads and boundary conditions a negative deflection may result somewhere in the plate. This indicates

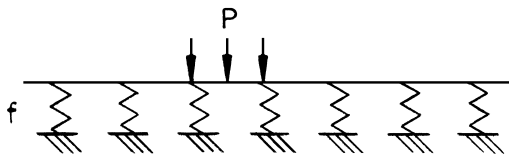


Figure 2-9. Plate on elastic foundation.

that the foundation must be able to sustain a tensile load at that location. This condition is very common in circular plates used in heat exchangers as discussed in Chapter 3 and the engineer must take appropriate precautions if the foundation cannot undergo a tensile force.

We can use the Levy method to obtain a solution of Eq. (2-9) for plates simply supported at two opposite edges with arbitrary boundary conditions at the other two edges. We proceed by solving the homogeneous and particular parts as in Eq. (1-42). Again expressing the deflection by the Fourier series

$$w_h = \sum_{m=1}^{\infty} f_m(y) \sin \frac{m\pi x}{a}$$

and solving the homogeneous part of Eq. (2-9), we get

$$w_h = \sum_{m=1}^{\infty} \left[A_m \sinh \alpha_m y \sin \beta_m y + B_m \sinh \alpha_m y \cos \beta_m y + C_m \cosh \alpha_m y \sin \beta_m y + D_m \cosh \alpha_m y \cos \beta_m y \right] \sin \frac{m\pi x}{a} \quad (2-10)$$

where

$$\alpha_m = \sqrt{\frac{1}{2} \left(\frac{m^2 \pi^2}{a^2} + \sqrt{\frac{m^4 \pi^4}{a^4} + \frac{k}{D}} \right)} \quad (2-11)$$

and

$$\beta_m = \sqrt{\frac{1}{2} \left(\frac{m^2 \pi^2}{a^2} - \sqrt{\frac{m^4 \pi^4}{a^4} + \frac{k}{D}} \right)}. \quad (2-12)$$

Similarly, if we express the deflection by

$$w_p = \sum_{m=1}^{\infty} g_m(y) \sin \frac{m\pi x}{a} \quad (2-13)$$

and the applied loads as

$$p = \sum_{m=1}^{\infty} p_m(y) \sin \frac{m\pi x}{a} \quad (2-14)$$

and we substitute these equations into Eq. (2-9) to get

$$\frac{d^4 g_m}{dy^4} - 2 \left(\frac{m\pi}{a} \right)^2 \frac{d^2 g_m}{dy^2} + \left(\frac{m^4 \pi^4}{a^4} + \frac{k}{D} \right) g_m = \frac{p_m(y)}{D}. \quad (2-15)$$

Equations (2-10) and (2-15) are the two governing expressions for rectangular plates on an elastic foundation with two simply supported edges.

Example 2-5

Find the expression for the deflection of the plate shown in Fig. 2-10 that is resting on a foundation of modulus k_0 and subjected to a uniform pressure p .

Solution

For a uniform pressure p , the particular solution of Eq. (2-15) gives

$$g_m = \frac{p}{D} \frac{4}{m\pi} \frac{1}{\left(\frac{m^4\pi^4}{a^4} + \frac{k_0}{D}\right)}. \quad (1)$$

Due to symmetry of the deflection around the x -axis, we can take $B_m = C_m = 0$ in Eq. (2-10) and the total deflection becomes

$$w_h = \sum_{m=1}^{\infty} \left[A_m \sinh \alpha_m y \sin \beta_m y + D_m \cosh \alpha_m y \cos \beta_m y + g_m \right] \sin \frac{m\pi x}{a}. \quad (2)$$

The boundary conditions are

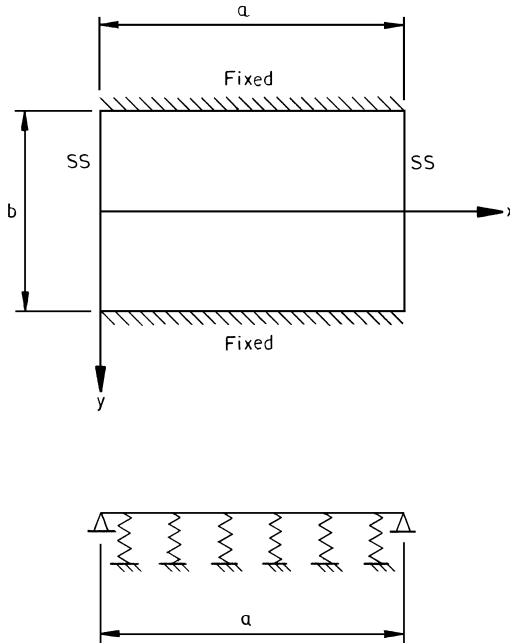


Figure 2-10. Plate on an elastic foundation with various boundary conditions.

$$\text{at } y = +b/2, \quad w = 0$$

and

$$\frac{\partial w}{\partial y} = 0.$$

Substituting these boundary conditions into Eq. (2) gives

$$A_m = \frac{g_m}{K_1 \left(\cosh \alpha_m \frac{b}{2} \cos \beta_m \frac{b}{2} \right)}$$

and

$$D_m = -A_m \tanh \alpha_m \frac{b}{2} \tan \beta_m \frac{b}{2} - \frac{g_m}{\cosh \alpha_m \frac{b}{2} \cos \beta_m \frac{b}{2}}$$

where

$$K_1 = \frac{K_2 + K_3}{K_4 - K_5} - \tanh \alpha_m \frac{b}{2} \tan \beta_m \frac{b}{2}$$

$$K_2 = \alpha_m \cosh \alpha_m \frac{b}{2} \sin \beta_m \frac{b}{2}$$

$$K_3 = \beta_m \sinh \alpha_m \frac{b}{2} \cos \beta_m \frac{b}{2}$$

$$K_4 = \alpha_m \sinh \alpha_m \frac{b}{2} \cos \beta_m \frac{b}{2}$$

$$K_5 = \beta_m \cosh \alpha_m \frac{b}{2} \sin \beta_m \frac{b}{2}.$$

With A_m and D_m known, Eq. (2) can now be solved for the deflection.

Problems

2-13 Find the expression for the deflection of a simply supported plate uniformly loaded and supported on an elastic foundation of modulus k_o .

2-14 Solve Example 1-3 if the plate is resting on an elastic foundation of modulus k_o .

2-4 Thermal Stress

A change in the temperature of a plate may result in a change in the length of the middle surface and a change in the curvature. Accordingly, the basic differential equations derived in Chapter 1 must be modified to consider temperature change.

We begin the derivation by defining u and v as the change in length of the middle surface in the x - and y -directions, respectively, of the plate shown in Figs. 1-3 and 1-4. Then Eq. (1-12) becomes

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ 0 \end{bmatrix} - z \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}. \tag{2-16}$$

Next we define the change in strain due to temperature change as $\alpha(\Delta T)$, where α is the coefficient of thermal expansion and ΔT is the temperature change which is a function of z through the thickness of the plate. Values of α for some commonly encountered materials are shown in Table 2-1.

For thin plates with temperature loads, Eq. (1-13) can be expressed as

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\mu & 0 \\ -\mu & 1 & 0 \\ 0 & 0 & 2(1 + \mu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} + \alpha(\Delta T) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{2-17}$$

Table 2-1. Coefficients of thermal expansion (multiplied by 10^6), in/in/°F

Material	Temperature, °F					
	Room Temperature	200	400	600	800	1000
Aluminum (6061)	12.6	12.9	13.5			
Brass (Cu-Zn)	9.6	9.7	10.2	10.7	11.2	11.6
Bronze (Cu-Al)				9.0 ^a		
Carbon steel	6.5	6.7	7.1	7.4	7.8	
Copper	9.4	9.6	9.8	10.1	10.3	10.5
Cu-Ni (70–30)	8.5	8.5	8.9	9.1		
Nickel alloy C276	6.1	6.3	6.7	7.1	7.3	
Nickel alloy 600	6.9	7.2	7.6	7.8	8.0	
Stainless steel	8.6	8.8	9.2	9.5	9.8	10.1
Titanium (Gr. 1,2)	4.7	4.7	4.8	4.9	5.1	
Zirconium alloys	3.2	3.4	3.7	4.0		
Concrete	5.5					
Wood, hard	2.7 ^b					
Wood, soft	3.6 ^b					

^aAt 500°F.
^bParallel to fibers.

or

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} - \frac{\alpha \Delta T E}{1-\mu} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}. \quad (2-18)$$

For the case where the plate bends without a change in the length of the middle surface, i.e., $u = v = 0$, we substitute Eqs. (2-16) and (2-18) into Eq. (1-16) and get

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = -D \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & -(1-\mu) \end{bmatrix} \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} - \frac{M_o}{1-\mu} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad (2-19)$$

where

$$M_o = \alpha E \int_{-t/2}^{t/2} (\Delta T) z \, dz. \quad (2-20)$$

Substituting Eqs. (2-19) and (2-20) into the plate equilibrium Eq. (1-24) gives

$$\nabla^4 w = p/D - \frac{1}{D(1-\mu)} \nabla^2 M_o. \quad (2-21)$$

Equation (2-21) is the governing differential equation for the bending of a rectangular plate due to lateral pressure and thermal loads.

For the case where $p = 0$, Eq. (2-21) reduces to

$$\nabla^2 w = -\frac{M_o}{D(1-\mu)} \quad (2-22)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Next we consider the case where the length of the middle surface changes due to temperature variation without any lateral deflection, i.e., $w = 0$. For this case we need, in addition to the forces shown in Fig. 1-6, three in-plane forces N_x , N_y , and N_{xy} as shown in Fig. 2-11. The stress-force relationship is expressed as

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz \quad (2-23)$$

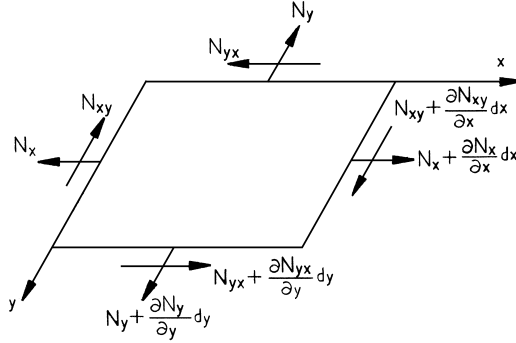


Figure 2-11. In-plane forces in a rectangular plate.

Substituting Eq. (2-18) into Eq. (2-23) gives

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1}{Et} \begin{bmatrix} 1 & -\mu & 0 \\ -\mu & 1 & 0 \\ 0 & 0 & 2(1+\mu) \end{bmatrix} \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} + \frac{N_o}{Et} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad (2-24)$$

where

$$N_o = \alpha E \int_{-t/2}^{t/2} (\Delta T) dz. \quad (2-25)$$

From Eq. (2-16) we observe that each of the three strains is a function of the deflections u , v , and w . Hence, a compatibility equation (Timoshenko and Goodier 1951) that combines the three strains is obtained from Eq. (2-16) and is expressed as

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}. \quad (2-26)$$

Substituting Eq. (2-24) into this expression yields the differential equation

$$\frac{\partial^2}{\partial y^2} (N_x - \mu N_y + N_o) + \frac{\partial^2}{\partial x^2} (-\mu N_x + N_y + N_o) - 2(1+\mu) \frac{\partial^2 N_{xy}}{\partial x \partial y} = 0. \quad (2-27)$$

In order to solve this equation, we need to investigate the in-plane forces in the plate. Summing forces (Fig. 2-11) in the x - and y -directions yields

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0 \quad (2-28)$$

and

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0.$$

These equations are satisfied by selecting a stress function $\psi(x, y)$ that is defined by

$$N_x = \frac{\partial^2 \psi}{\partial y^2}, \quad N_y = \frac{\partial^2 \psi}{\partial x^2}, \quad \text{and} \quad N_{xy} = -\frac{\partial^2 \psi}{\partial x \partial y}. \quad (2-29)$$

Substituting Eq. (2-29) into Eq. (2-27) results in the differential equation

$$\nabla^4 \psi + \nabla^2 N_o = 0. \quad (2-30)$$

Equations (2-21) and (2-30) constitute the general solution of a plate subjected to temperature change. Equation (2-21) is solved by the methods discussed in this Chapter while Eq. (2-30) is solved by methods discussed in the theory of elasticity for plane stress problems which are beyond the scope of this book.

Example 2-6

Find the deflection in a simply supported plate due to decrease in temperature of the top surface of T_o and increase of the bottom surface by T_o .

Solution

$$T = T_o(2z/t)$$

and Eq. (2-20) becomes

$$M_o = \alpha E T_o t^2 / 6. \quad (1)$$

Let

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (2)$$

and

$$M_o = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} T_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (3)$$

where

$$T_{mn} = \frac{4}{ab} \int_0^b \int_0^a M_o \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy. \quad (4)$$

Substituting Eqs. (2) and (3) into Eq. (2-22) results in

$$w_{mn} = \frac{T_{mn}}{D(1-\mu)\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)}. \quad (5)$$

Substituting Eq. (1) and (4) into Eq. (5) gives

$$w_{mn} = \frac{8\alpha E T_o t^2}{3mnD(1-\mu)\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)} \quad (6)$$

where m and n are odd.

Since the temperature variation does not affect the middle surface, i.e., $u = v = 0$, Eq. (2-30) is redundant and need not be considered.

Problems

2-15 Calculate the maximum moment in Example 2-6.

2-16 Find the expression for the bending moment in the plate shown in Fig. P2-5 due to the temperature variation given by

$$T = T_0(2z/t).$$

Hint: Use the solution of Problem P2-6 to satisfy the boundary conditions.

2-5 Design of Various Rectangular Plates

The maximum bending and deflection of rectangular plates with various boundary conditions have been solved and tabulated in many references.

Szilard (1974) as well as the references given in Section 1-9 tabulate maximum moments and deflections for rectangular plates with some commonly encountered boundary conditions. Moody (1970) lists numerous moment tables for rectangular plates free at one edge with various boundary conditions at the remaining edges and subjected to various loading conditions. Continuous plates are designed in accordance with the equations developed in Section 2-2.

The ASME VIII-1 code contains rules for the design of jacketed cylindrical and spherical shells, Fig. 2-12, that are based on the stayed plate theory and Eq. (2-7). ASME conservatively assumes the shells to have large radius of curvature to thickness ratios where the behavior of the shell approaches that of

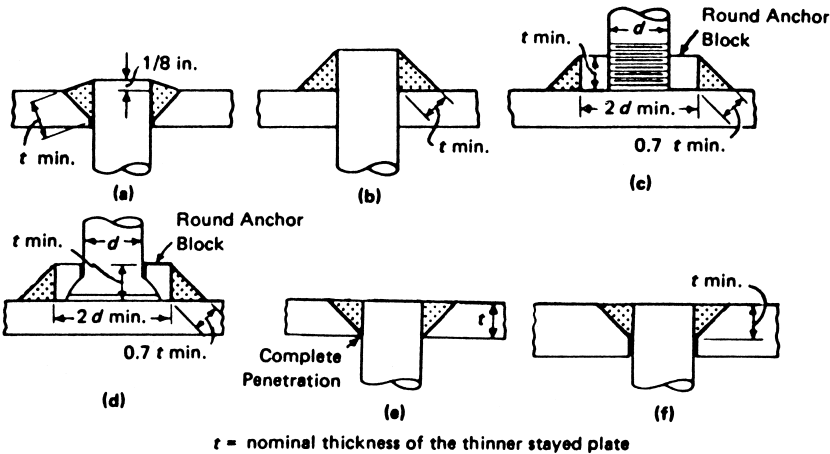


Figure 2-12. Welded staybolts. (Courtesy of ASME.)

a flat plate. Letting $\mu = 0.3$ and using a b/a ratio of 0.4, Eq. (2-7) can be reduced to

$$t = a \sqrt{\frac{p}{2.28\sigma}} \quad (2-31)$$

The value of 2.28 in the denominator is varied by the ASME from 2.1 to 3.5 depending on the type of construction and method of weld attachment.

For continuous concrete slabs supported by columns without intermediate beams, the reinforcement cutoffs are also based on Eq. (2-7) and are detailed in the ACI 318 code.

Allowable stress values for some materials were given in Table 1-1 at temperatures below the creep and rupture range as defined by ASME VIII-1. For high-temperature applications, the ASME criteria in the creep and rupture range are based on limiting the allowable stress to the lower of the following values:

1. 100% of the average stress for a creep rate of 0.01%/100 hr.
2. 67% of the average stress for rupture at the end of 100,000 hr.
3. 80% of the minimum stress for rupture at the end of 100,000 hr.

Using these criteria, the allowable stress values for the materials listed in table 1-1 that the permitted at high temperatures are shown in Table 2-2.

Table 2-2. ASME VIII-1 allowable stress values at elevated temperature, ksi

Material	ASME Designation	Temperature, °F		
		900	1100	1300
Carbon steel	SA 516-70	6.7		
Stainless steel	SA 240-304	14.6	9.8	3.7
Nickel alloy	SB 575-276	22.6	15.0	

Problems

2-17 The inner and outer shells of a pressure vessel are stayed together on a 12-inch stay pitch. The pressure between the cylinders is 50 psi. Use Eq. (2-31) to determine the required thickness of the cylinders. Disregard the hoop stress in the cylinders due to pressure since it is small in magnitude for most applications. Let the allowable bending stress be equal to 15 ksi.

2-18 Determine the required diameter of the stays in Problem 2-17. Let the allowable tensile stress = 20 ksi. If the stays are attached as shown in Fig. 2-12b, calculate the required size of the fillet welds. The allowable stress in shear = 12 ksi.

Bending of Circular Plates

3-1 Plates Subjected to Uniform Loads in the θ -Direction

Circular plates are common in many structures such as nozzle covers, end closures in pressure vessels, and bulkheads in submarines and airplanes. The derivation of the classical equations for lateral bending of circular plates dates back to 1828 and is accredited to Poisson (Timoshenko 1983). He used polar coordinates to transfer the differential equations for the bending of a rectangular plate to circular plates. The first rigorous solution of the differential equation of circular plates for various loading and boundary conditions was made around 1900 and is credited to A. E. H. Love (Love 1944).

The basic assumptions made in deriving the differential equations for lateral bending of rectangular plates in Section 1-1 are also applicable to circular plates. The differential equations for the lateral bending of circular plates subjected to uniform loads in the θ -direction are derived from Fig. 3-1. For sign convention it will be assumed that downward deflections and clockwise rotations are positive. Hence, if a flat plate undergoes a small deflection as shown in Fig. 3-1, then the radius of curvature r at point B is given by

$$\sin(\phi) \approx \phi = r/r_0$$

or

$$\frac{1}{r_0} = \phi/r = \frac{-1}{r} \frac{dw}{dr}. \quad (3-1)$$

The quantity r_0 represents a radius that forms a cone as it rotates around the z -axis (in and out of the plane of the paper). The second radius of curvature is denoted by r_r . The origin of r_r does not necessarily fall on the axis of symmetry although, for any point B , the radii r_r and r_0 coincide with each other. The value of r_r is obtained from Eq. (1-6) as

$$\chi = \frac{1}{r_r} = -\frac{d^2w}{dr^2}$$

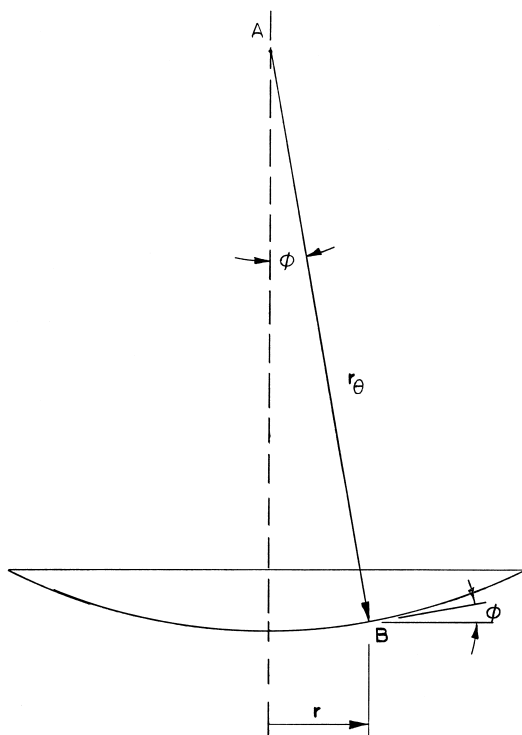


Figure 3-1. Lateral bending of circular plate.

or

$$\chi = \frac{1}{r_r} = -\frac{d^2 w}{dr^2} = \frac{d\phi}{dr}. \quad (3-2)$$

The M_x and M_y expressions in Eq. (1-17) can be written in terms of the radial and tangential directions as

$$\begin{bmatrix} M_r \\ M_t \end{bmatrix} = D \begin{bmatrix} 1 & \mu \\ \mu & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{r_r} \\ \frac{1}{r_\theta} \end{bmatrix} \quad (3-3)$$

or

$$\begin{bmatrix} M_r \\ M_t \end{bmatrix} = D \begin{bmatrix} 1 & \frac{\mu}{r} \\ \mu & \frac{1}{r} \end{bmatrix} \begin{bmatrix} \frac{d\phi}{dr} \\ \phi \end{bmatrix} \quad (3-4)$$

and

$$\begin{bmatrix} M_r \\ M_t \end{bmatrix} = -D \begin{bmatrix} 1 & \frac{\mu}{r} \\ \mu & \frac{1}{r} \end{bmatrix} \begin{bmatrix} \frac{d^2 w}{dr^2} \\ \frac{dw}{dr} \end{bmatrix} \quad (3-5)$$

where

$$D = \frac{Et^3}{12(1 - \mu^2)}.$$

The classical theory of the lateral bending of circular plates discussed in this section is based on the assumption that the loads on the plate are uniformly distributed in the θ direction. In this case the torsional moment $M_{r\theta}$ is zero and the other forces are as shown in Fig. 3-2a. Summing moments around line a-a gives

$$\begin{aligned} (M_r r d\theta) - \left(M_r + \frac{dM_r}{dr} dr \right) (r + dr) d\theta + 2(M_t dr d\theta/2) \\ - Q r d\theta dr/2 - \left(Q + \frac{dQ}{dr} dr \right) (r + dr) d\theta dr/2 = 0. \end{aligned} \quad (3-6)$$

The quantity $M_t dr d\theta/2$ is the component of M_t perpendicular to axis a-a as shown in Fig. 3-2b. Equation (3-6) can be reduced to

$$M_r + \frac{dM_r}{dr} r - M_t + Qr = 0. \quad (3-7)$$

Substituting Eq. (3-5) into this equation gives

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dw}{dr} \right) \right] = Q/D. \quad (3-8)$$

Or, since

$$2\pi r Q = \int p 2\pi r dr,$$

Eq. (3-8) can be written in a different form as

$$\frac{1}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dw}{dr} \right) \right] \right\} = \frac{p}{D} \quad (3-9)$$

where p is a function of r .

The analysis of circular plates with uniform thickness subjected to symmetric lateral loads consists of solving the differential equation for the deflection as given by Eq. (3-8) or (3-9). The bending moments are then calculated from Eq. (3-5). The shearing force is calculated from Eqs. (3-7) and (3-5) as

$$Q = D \left(\frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} \right) \quad (3-10)$$

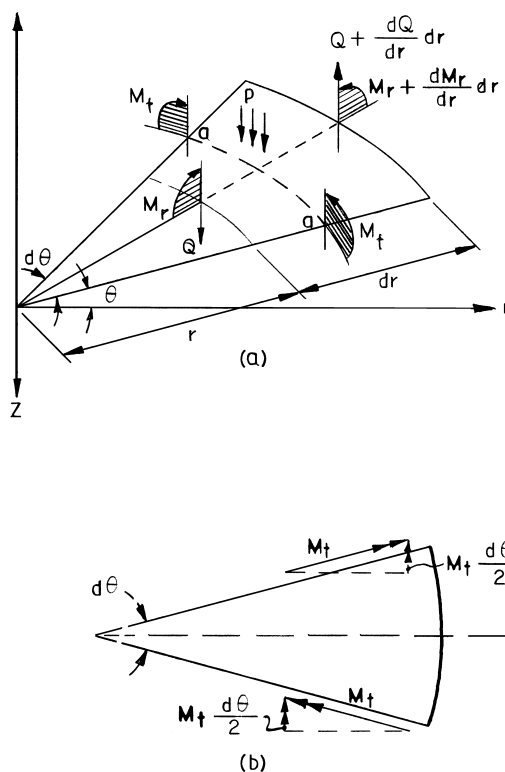


Figure 3-2. Forces and moments in a circular plate symmetrically loaded in the θ -direction.

or, from Eqs. (3-7) and (3-4), as

$$Q = -D \left(\frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} - \frac{\phi}{r^2} \right). \quad (3-11)$$

Example 3-1

- Find the expression for the maximum moment and deflection of a uniformly loaded circular plate with simply supported edges.
- Find the required thickness of a steel plate if the allowable stress = 15,000 psi, $p = 5$ psi, $a = 20$ inches, $E = 30,000,000$ psi, and $\mu = 0.3$. What is the maximum deflection?
- For a concrete plate, μ is usually taken as zero. What are the moment expressions at $r = 0$, $r = a/2$, and $r = a$?

Solution

- From Fig. 3-3, the shearing force Q at any radius r is given by

$$2\pi r Q = \pi r^2 p$$

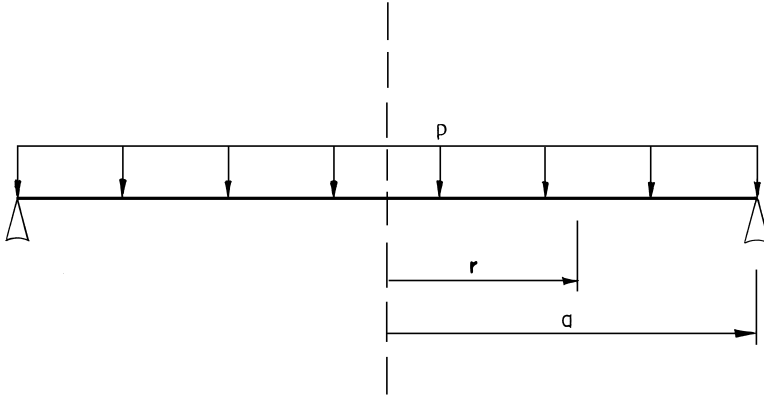


Figure 3-3. Uniformly loaded plate.

or

$$Q = pr/2.$$

From Eq. (3-8)

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dw}{dr} \right) \right] = \frac{pr}{2D}.$$

Integrating this equation gives

$$\text{slope} = \frac{dw}{dr} = \frac{pr^3}{16D} + \frac{C_1 r}{2} + \frac{C_2}{r} \quad (1)$$

$$\text{deflection} = w = \frac{pr^4}{64D} + \frac{C_1 r^2}{4} + C_2 \ln r + C_3. \quad (2)$$

At $r = 0$ the slope is equal to zero due to symmetry. Hence, from Eq. (1), C_2 must be set to zero. At $r = a$, the moment is zero and

$$-D \left(\frac{d^2 w}{dr^2} + \frac{\mu}{r} \frac{dw}{dr} \right) = 0$$

or

$$C_1 = \frac{-(3 + \mu)}{(1 + \mu)} \frac{pa^2}{8D}.$$

At $r = a$, the deflection is zero and Eq. (2) gives

$$C_3 = \frac{pa^4}{64D} \left(\frac{6 + 2\mu}{1 + \mu} - 1 \right).$$

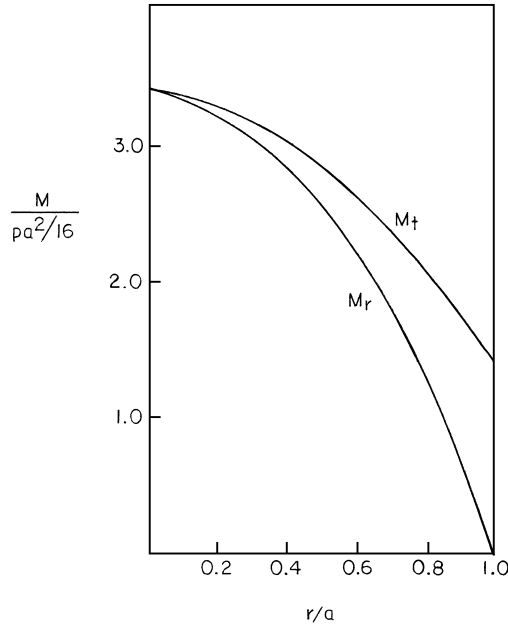


Figure 3-4. Plot of M_r and M_t of a uniformly loaded plate.

The expression for deflection becomes

$$w = \frac{p}{64D} (a^2 - r^2) \left(\frac{5 + \mu}{1 + \mu} a^2 - r^2 \right).$$

Substituting this expression into Eq. (3-5) gives

$$M_r = \frac{p}{16} (3 + \mu) (a^2 - r^2)$$

$$M_t = \frac{p}{16} [a^2(3 + \mu) - r^2(1 + 3\mu)].$$

The maximum deflection is at $r = 0$ and is given by

$$\max w = \frac{pa^4}{64D} \left(\frac{5 + \mu}{1 + \mu} \right).$$

A plot of M_r and M_t is shown in Fig. 3-4 for $\mu = 0.3$. The plot shows that the maximum moment is in the center and is equal to

$$M_r = M_t = \frac{pa^2}{16} (3 + \mu).$$

It is of interest to note that M_t is not zero at the edge of the plate. This is important in reinforced concrete plates as reinforcing bars are needed around the perimeter to resist the tension stress caused by M_t .

(b)

$$\text{Maximum } M = \frac{5 \times 20^2}{16} (3.3) = 412.5 \text{ inch-lbs/inch}$$

and

$$t = \sqrt{\frac{6M}{\sigma}}$$

$$t = \sqrt{\frac{6 \times 412.5}{15,000}}$$

$$= 0.41 \text{ inch.}$$

$$\text{Maximum } w = \frac{pa^4}{64D} \left(\frac{5 + \mu}{1 + \mu} \right) = 0.27 \text{ inch}$$

(c) For $\mu = 0$, the moment expressions become

$$M_r = \frac{3p}{16} (a^2 - r^2)$$

$$M_t = \frac{p}{16} (3a^2 - r^2).$$

At $r = 0$,

$$M_r = M_t = \frac{3pa^2}{16}.$$

At $r = a/2$,

$$M_r = \frac{9pa^2}{64}, \quad M_t = \frac{11pa^2}{64}.$$

At $r = a$,

$$M_r = 0 \quad M_t = \frac{pa^2}{8}.$$

Figure 3-5 shows a general layout of reinforcing bars in a circular concrete slab.

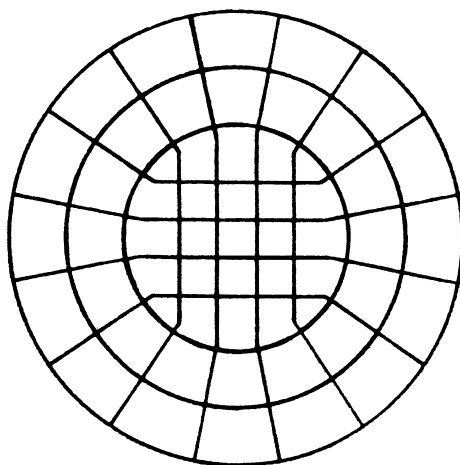
Example 3-2

Find the stress at $r = a$ and $r = b$ for the plate shown in Fig. 3-6. Let $a = 24$ inches, $b = 12$ inches, $F = 20$ lbs/inch, $t = 0.50$ inch, $E = 30,000$ ksi, and $\mu = 0.3$.

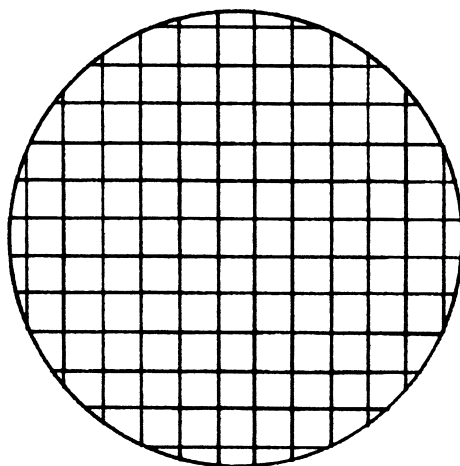
Solution

The shearing force at any point is given by

$$Q = bF/r.$$



(a) radial and circular pattern



(b) square mesh pattern

Figure 3-5. Layout of reinforcing bars in a circular plate.

Substituting this expression into Eq. (3-8) and integrating results in

$$w = \frac{bF}{4D} r^2 (\ln r - 1) + \frac{C_1 r^2}{4} + C_2 \ln r + C_3. \quad (1)$$

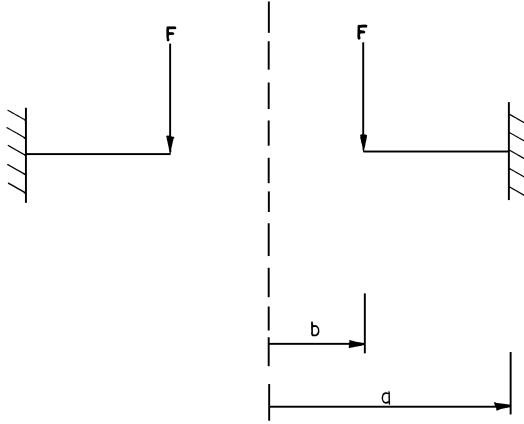


Figure 3-6. Plate loaded at its inner edge.

The boundary conditions are

$$\begin{aligned} w &= 0 & \text{at } r &= a \\ dw/dr &= 0 & \text{at } r &= a \\ M_r &= 0 & \text{at } r &= b. \end{aligned}$$

Evaluating Eq. (1) and its derivatives at the boundary conditions results in

$$\begin{aligned} C_1 &= -648.53/D \\ C_2 &= 1667.46/D \\ C_3 &= 12,814.46/D. \end{aligned}$$

At $r = b$,

$$\begin{aligned} M_r &= 0 \\ M_t &= -D \left(\frac{1}{r} \frac{dw}{dr} + \mu \frac{d^2w}{dr^2} \right) \\ &= -D \left\{ \frac{bF}{2D} \ln r - \frac{bF}{2D} \frac{1}{2} \right. \\ &\quad \left. + \mu \left[\frac{bF}{2D} \left(\frac{1}{2} + \ln r \right) + \frac{C_1}{2} - \frac{C_2}{r^2} \right] \right\} \end{aligned}$$

or

$$M_t = -67.8 \text{ inch-lbs/inch}$$

and

$$\sigma_t = 6M/t^2 = 1627 \text{ psi.}$$

Similarly, at $r = a$

$$M_t = -34.26 \text{ inch-lbs/inch}$$

$$\sigma_t = 822 \text{ psi}$$

$$M_r = -114.21 \text{ inch-lbs/inch}$$

$$\sigma_r = 2741 \text{ psi.}$$

Example 3-3

Find the expression for the deflection of the plate shown in Fig. 3-7a due to load F .

Solution

The plate can be separated into two components (Fig. 3-7b). Continuity between the inner and the outer plate is maintained by applying an unknown moment, M_o .

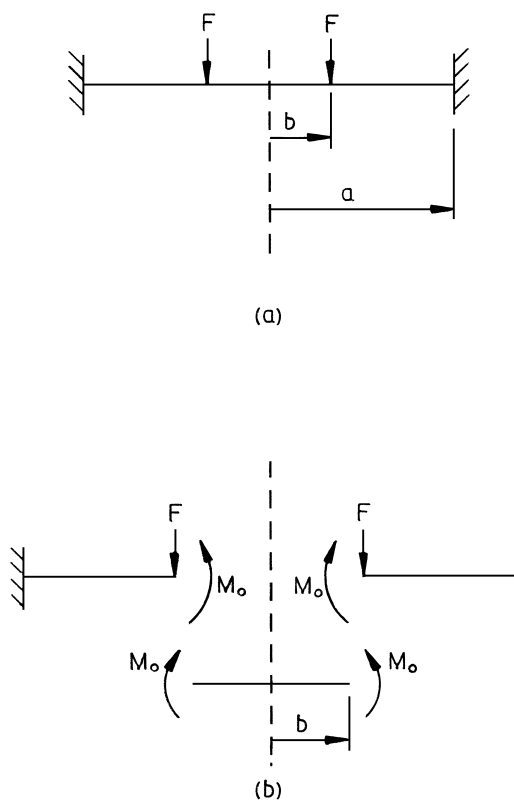


Figure 3-7. Plate loaded at distance b from centerline.

as shown in Fig. 3-7b. The deflection of the inner plate due to M_o is obtained from Eq. (3-8) with $Q = 0$ as

$$w = \frac{C_1 r^2}{4} + C_2 \ln r + C_3. \quad (1)$$

The slope is

$$\frac{dw}{dr} = \frac{C_1 r}{2} + \frac{C_2}{r}, \quad (2)$$

and M_r is obtained from Eq. (3-5) as

$$M_r = -D \left[\frac{C_1}{2} (1 + \mu) - \frac{C_2}{r^2} (1 - \mu) \right]. \quad (3)$$

At $r = 0$, the slope is zero and from Eq. (2) we get $C_2 = 0$.

At $r = b$, $M_r = M_o$ and Eq. (3) yields

$$C_1 = \frac{-2M_o}{D(1 + \mu)}. \quad (4)$$

Equations (1) and (2) can be expressed as

$$w = \frac{-M_o r^2}{2D(1 + \mu)} + C_3 \quad (5)$$

$$\frac{dw}{dr} = \frac{-rM_o}{D(1 + \mu)}. \quad (6)$$

The deflection of the outer plate is obtained from Example 3-2 as

$$w = \frac{bF}{4D} r^2 (\ln r - 1) + \frac{C_4 r^2}{4} + C_5 \ln r + C_6. \quad (7)$$

At $r = a$, the slope is zero and Eq. (7) gives

$$\frac{C_4 a}{2} + \frac{C_5}{a} = \frac{-bFa}{4D} (2 \ln a - 1). \quad (8)$$

At $r = b$, $M_r = M_o$ and from Eqs. (7) and (3-5) we get

$$\frac{-M_o}{D} = \frac{bF}{4D} [(1 - \mu) + 2(1 + \mu) \ln b] + \frac{C_4}{2} (1 + \mu) + \frac{C_5}{b^2} (\mu - 1). \quad (9)$$

At $r = b$, the slope of the outer plate is equal to the slope of the inner plate. Taking the derivatives of Eq. (7) and equating it to Eq. (6) at $r = b$ gives

$$\frac{C_4 b}{2} + \frac{C_5}{b} = \frac{-b^2 F}{4D} (2 \ln b - 1) - \frac{bM_o}{D(1 + \mu)}. \quad (10)$$

Equations (8), (9), and (10) contain three unknowns. They are M_o , C_4 , and C_5 . Solving these three equations yields

$$M_o = \frac{(1 + \mu)bF}{4a^2} [2a^2 \ln(a/b) - (a^2 - b^2)] \quad (11)$$

$$C_4 = \frac{bF}{D} \left(\frac{a^2 - b^2}{2a^2} - \ln a \right) \quad (12)$$

$$C_5 = \frac{b^3 F}{4D}. \quad (13)$$

With these quantities known, the other constants can readily be obtained. Constant C_1 is determined from Eq. (4). Constant C_6 is solved from Eq. (7) for the boundary conditions $w = 0$ at $r = a$. This gives

$$C_6 = \frac{bF}{8D} (a^2 + b^2 - 2b^2 \ln a). \quad (14)$$

Constant C_3 can now be calculated from the equation

$$w \text{ of inner plate}|_{r=b} = w \text{ of outer plate}|_{r=b}.$$

Equating Eqs. (1) and (7) at $r = b$ gives

$$C_3 = \frac{bF}{8a^2D} [-2a^2b^2 \ln(a/b) + (a^2 - b^2)a^2]. \quad (15)$$

Hence, the deflection of the inner plate is obtained by substituting Eqs. (11) and (15) into Eq. (5) to give

$$w = \frac{bF}{8a^2D} [(a^2 - b^2)(a^2 + r^2) - 2a^2(b^2 + r^2) \ln(a/b)]$$

and the deflection of the outer plate is obtained by substituting Eqs. (12), (13), and (14) into Eq. (7). This gives

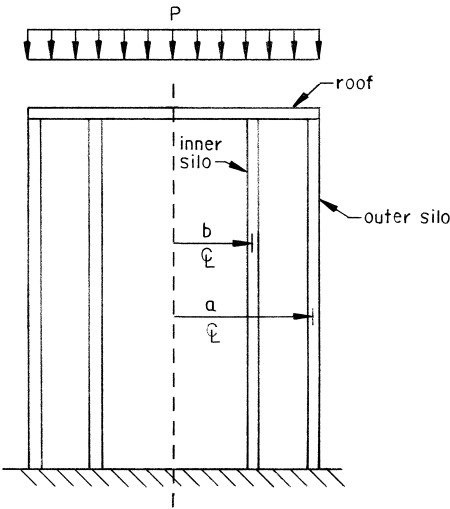
$$w = \frac{bF}{8a^2D} [(a^2 + b^2)(a^2 - r^2) + 2a^2(b^2 + r^2) \ln(r/a)].$$

Problems

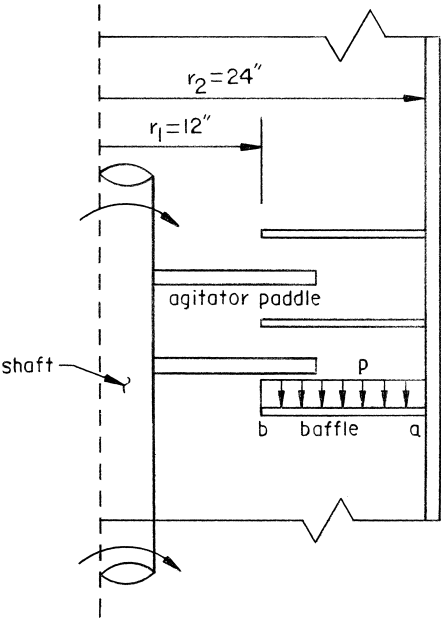
3-1 The double concrete silo is covered by a concrete flat roof as shown. Find the moments in the roof due to an applied uniform load p , and draw the M_r and M_t diagrams. The attachment of the roof to the cylindrical silos is assumed simply supported. Let $\mu = 0$.

3-2 Stainless steel baffles are attached to a vessel that has an agitator shaft. The attachment of the baffles to the vessel is assumed fixed and the uniform pressure due to agitator rotation is 20 psi. What are the maximum values of M_r and M_t and where do they occur? What is the maximum deflection at point b ? Let $E = 27,000,000$ psi and $\mu = 0.29$. Also, if the baffles are assumed as fixed cantilevered beams, what will the maximum moment be and how does it compare to M_r and M_t ?

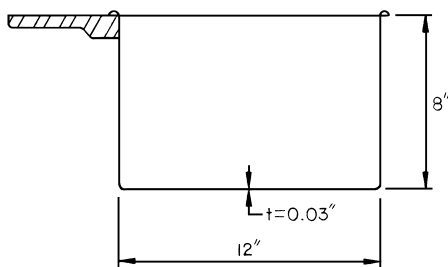
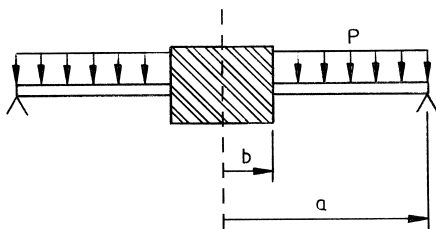
3-3 A pan is made of aluminum and is full of water. If the edge of the bottom plate is assumed fixed, what is the maximum stress due to the exerted water pressure? Let $\gamma = 62.4$ pcf, $t = 0.030$ inch, $E = 10,200$ ksi, and $\mu = 0.33$. What is the maximum deflection?



Prob. 3-1. Double concrete silo.



Prob. 3-2. Baffles in a cylindrical vessel.

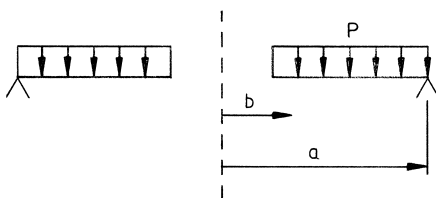
**Prob. 3-3.** Aluminum pan.**Prob. 3-5.** Plate with rigid central attachment.

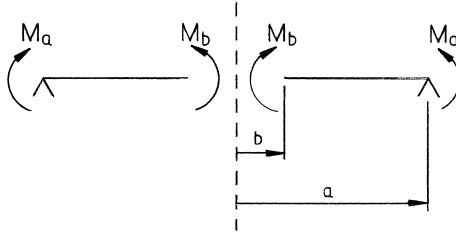
3-4 The pan in Problem 3-3 is empty and is at a temperature of 100°F . What is the thermal stress in the bottom plate if the bottom surface of the bottom plate is subjected to a temperature of 160°F and the top surface is subjected to a temperature of 40°F ? Let the coefficient of expansion be 13.5×10^{-6} inches/inch/ $^{\circ}\text{F}$.

3-5 Find the expressions for the moments in the circular plate.

3-6 Find the expressions for the moments in the circular plate.

3-7 Find the expressions for the moments in the circular plate.

**Prob. 3-6.** Plate with central hole.



Prob. 3-7. Plate with inner and outer moments.

3-2 Plates with Variable Thickness and Subjected to Uniform Loads in the θ -Direction

Circular plates with variable thickness are encountered in many machine parts such as turbine blades, bellows, and springs. The analysis of such plates (Szilard 1974) is similar to that of plates with constant thickness except that the flexural rigidity D is a variable rather than a constant. Substituting Eq. (3-4) into the differential Eq. (3-7) gives

$$D \frac{d}{dr} \left(\frac{d\phi}{dr} + \frac{\phi}{r} \right) + \frac{dD}{dr} \left(\frac{d\phi}{dr} + \mu \frac{\phi}{r} \right) = -Q \quad (3-12)$$

or

$$D \frac{d^2\phi}{dr^2} + \left(\frac{D}{r} + \frac{dD}{dr} \right) \frac{d\phi}{dr} + \left(\mu \frac{dD}{dr} - \frac{D}{r} \right) \frac{\phi}{r} = -Q \quad (3-13)$$

where

$$\phi = -\frac{dw}{dr}.$$

For a uniformly loaded plate,

$$Q = \frac{1}{2\pi r} \int p(2\pi r) dr = pr/2.$$

Defining

$$\rho = r/a$$

where

a = outer radius of plate,

Eq. (3-13) becomes

$$D \frac{d^2\phi}{d\rho^2} + \left(\frac{D}{\rho} + \frac{dD}{d\rho} \right) \frac{d\phi}{d\rho} + \left(\mu \frac{dD}{d\rho} - \frac{D}{\rho} \right) \frac{\phi}{\rho} = \frac{-p\rho a^3}{2}. \quad (3-14)$$

The solution of Eq. (3-14) depends on specifying an expression for the thickness t . A commonly encountered class of plates is shown in Fig. 3-8. Solution of the plate shown in Fig. 3-8a is obtained by defining

$$t = Kr$$

and

$$D = \frac{EK^3r^3}{12(1-\mu^2)}.$$

Equation (3-14) becomes

$$\rho^3 \frac{d^2\phi}{d\rho^2} + 4\rho^2 \frac{d\phi}{d\rho} - (1-3\mu)\rho\phi = \frac{-12Q(1-\mu^2)}{EK^3a}. \quad (3-15)$$

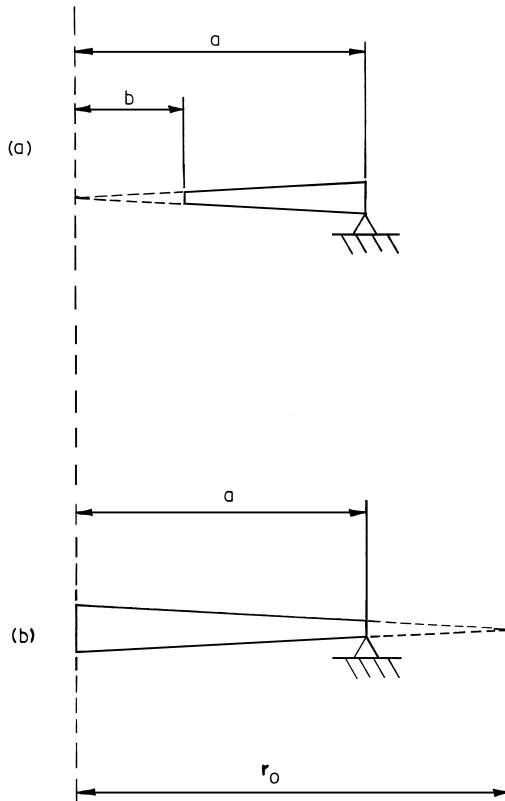


Figure 3-8. Plate with variable thickness.

The homogeneous solution of Eq. (3-15) can be expressed as

$$\phi_h = A\rho^a + B\rho^b$$

where

$$a = [-3 + \sqrt{9 - 4(3\mu - 1)}]/2$$

$$b = [-3 - \sqrt{9 - 4(3\mu - 1)}]/2$$

and A and B are constants.

Solution of the plate shown in Fig. 3-8b is obtained by defining

$$t = t_0(1 - r/r_0)$$

and

$$D = D_0(1 - \rho)^3$$

where,

$$D_0 = \frac{Et_0^3}{12(1 - \mu^2)} \quad \text{and} \quad \mu = 1/3,$$

and Eq. (3-14) becomes

$$\rho^2(1 - \rho)^3 \frac{d^2\phi}{d\rho^2} + (1 - 4\rho)(1 - \rho)^2 \frac{d\phi}{d\rho} - (1 - \rho)^2\phi = \frac{Qr^2\rho^2}{D_0}. \quad (3-16)$$

The homogeneous solution of Eq. (3-16) is given by

$$\phi_h = A\left(\frac{1 + 2\rho}{\rho}\right) + B\left(\frac{3\rho - 2\rho^2}{(1 - \rho)^2}\right)$$

where A and B are constants.

The particular solution of Eqs. (3-15) and (3-16) is obtained once the applied loads are defined.

Example 3-4

Determine the expression for the deflection in the plate shown in Fig. 3-9. Let $\mu = 1/3$.

Solution

Let

$$t = Cr.$$

The homogeneous solution of Eq. (3-15) becomes

$$\phi_h = A + B\rho^{-3}.$$

To solve for a particular solution, let $Q = \frac{F}{2\pi a\rho}$
and

$$\phi_p = \frac{G}{\rho^2}.$$

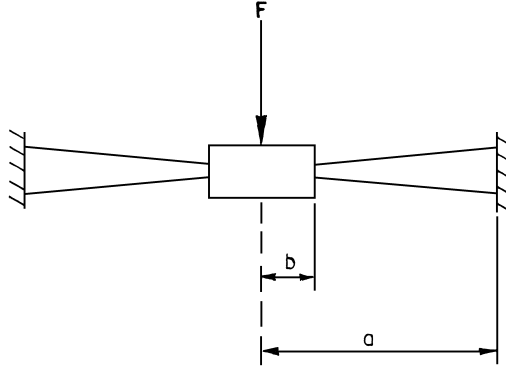


Figure 3-9. Plate with variable thickness and concentrated load at center.

Substituting into Eq. (3-15) gives

$$G = \frac{3(1 - \mu^2)F}{\pi EC^3 a^2}$$

and the total solution becomes

$$\phi = A + B\rho^{-3} + \frac{G}{\rho^2}. \quad (1)$$

The boundary conditions are

$$\text{at } r = b, \quad \rho = b/a \quad \text{and} \quad \phi = 0$$

and

$$\text{at } r = a, \quad \rho = 1 \quad \text{and} \quad \phi = 0.$$

Hence, from Eq. (1)

$$A = G \frac{a^2(b - a)}{a^3 - b^3}$$

$$B = G \frac{b(a^2 - b^2)}{a^3 - b^3}.$$

Integrating Eq. (1) to obtain w , and solving for the boundary condition

$$w = 0 \quad \text{at} \quad \rho = 1$$

results in the following expression for the deflection

$$w = Aa(\rho - 1) + \frac{Ba}{2} \left(1 - \frac{1}{\rho^2} \right) + Ga \left(1 - \frac{1}{\rho} \right).$$

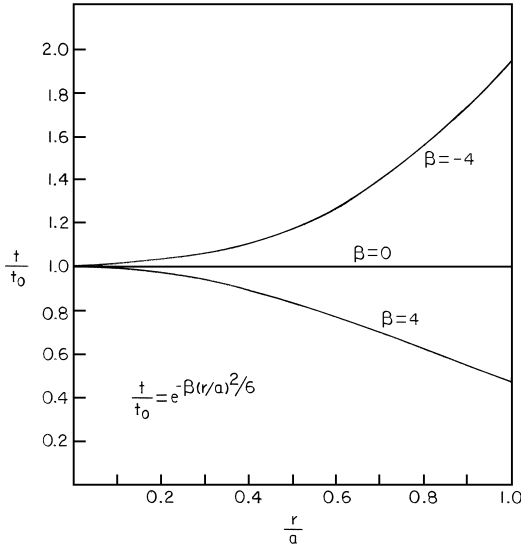


Figure 3-10. Thickness t as a function of β .

Another class of problems that is often encountered in machine parts is plates with non-linear variable thickness. The variable thickness can be expressed as (Timoshenko and Woinowsky-Krieger 1959)

$$t = t_0 e^{-\beta \rho^2/6} \quad (3-17)$$

where

β = factor defining thickness of plate as shown in Fig. 3-10;

t_0 = thickness of plate at center.

Substituting Eq. (3-17) into Eq. (3-14) gives

$$\frac{d^2\phi}{d\rho^2} + \left(\frac{1}{\rho} - \beta\rho\right) \frac{d\phi}{d\rho} - \left(\frac{1}{\rho^2} + \mu\beta\right) \phi = -K\rho e^{\beta\rho^2/2} \quad (3-18)$$

where

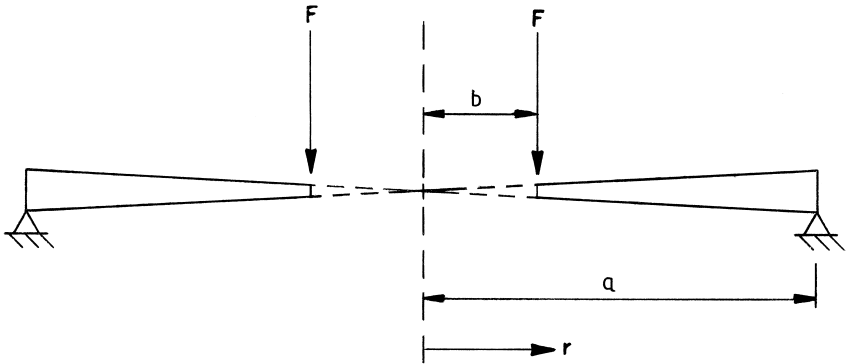
$$K = \frac{6(1-\mu^2)a^3p}{Et_0^3}.$$

The solution of Eq. (3-18) is given by

$$\phi = \phi_h + \phi_p.$$

Let the particular solution be expressed as

$$\phi_p = A\rho e^{\beta\rho^2/2} + B e^{\beta\rho^2/2}.$$



Prob. 3-9. Plate with variable thickness loaded at the inner edge.

Substituting this expression into Eq. (3-18) gives

$$\phi_p = \frac{-K}{(3-\mu)\beta} \rho e^{\beta \rho^2/2}.$$

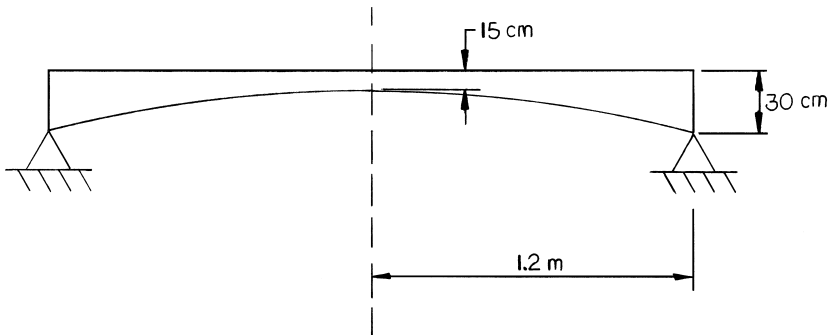
Pirchler (Timoshenko and Woinowsky-Krieger 1959) suggested a homogeneous solution in terms of a series of the form

$$\phi_h = A_1 \left(\rho + \sum_{m=1}^{\infty} \frac{(1+\mu)(3+\mu)\dots(2m-1+\mu)}{2.4.6.6\dots 2m.2m.(2m+2)} \rho^{2m+1} \right)$$

and the total solution is given by

$$\phi = K \left(C \phi_h - \frac{\rho}{(3-\mu)\beta} e^{\beta \rho^2/2} \right). \quad (3-19)$$

The constant C is obtained from the boundary condition of a solid plate and the maximum moments are determined from Eq. (3-19).



Prob. 3-10. Concrete plate.

Problems

3-8 Find the expression for the bending moments M_r and M_t for the plate shown in Fig. 3-8. Let $\mu = 1/3$.

3-9 Find the expression for the bending moments M_r and M_t for the plate. Let $\mu = 1/3$.

3-10 Find the expression for M_r and M_t for the concrete circular plate. Let $p = 1000 \text{ kgf/m}^2$, $\beta = 4.16$, and $\mu = 0$.

3-3 Plates Subjected to Nonuniform Loads in the θ -Direction

Many circular structures are encountered where the load distribution is variable in the θ -direction. These include stack foundations, submerged bulkheads, and nozzle covers subjected to connecting piping loads. The easiest method for deriving the governing equation for such problems is from the differential equation of rectangular plates. The reason is that the equation for rectangular plates includes the effects of torsional moments that are ignored in the derivation of the equation for circular plates with symmetric loading in the θ -direction and which are needed for the case where the load is variable in the θ -direction. The differential expression for rectangular plates is given by Eq. (1-25) as

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p(x, y)}{D}. \quad (3-20)$$

This equation must now be transferred to polar coordinates. Referring to Fig. 3-11,

$$r^2 = x^2 + y^2, \quad x = r \cos \theta, \quad y = r \sin \theta, \quad \tan \theta = y/x.$$

Hence

$$\frac{\partial r}{\partial x} = x/r = \cos \theta \quad \frac{\partial r}{\partial \theta} = y/r = \sin \theta$$

$$\frac{\partial}{\partial x} \tan \theta = \frac{\partial}{\partial x} (y/x)$$

$$\frac{1}{\cos^2 \theta} \frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x} (y/x)$$

$$\frac{\partial \theta}{\partial x} = -y \frac{\cos^2 \theta}{x^2} = -y/r^2 = -\sin \theta / r$$

$$\frac{\partial \theta}{\partial y} = x/r^2 = \cos \theta / r.$$

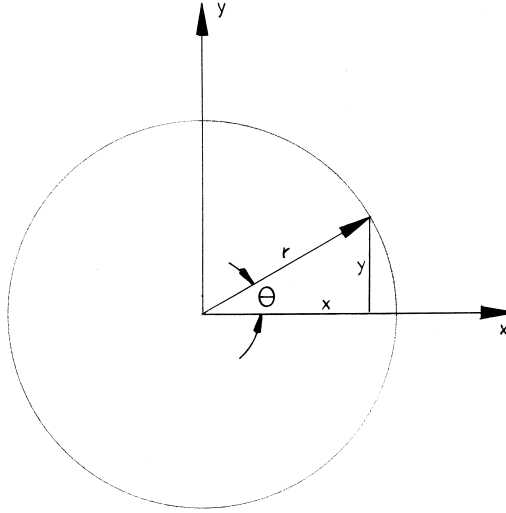


Figure 3-11. Coordinate system.

Using the chain rule of partial derivatives,

$$\begin{aligned}
 \frac{\partial w}{\partial x} &= \frac{\partial w}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial w}{\partial \theta} \frac{\partial \theta}{\partial x} \\
 &= \frac{\partial w}{\partial r} \cos \theta - \frac{1}{r} \frac{\partial w}{\partial \theta} \sin \theta \\
 \frac{\partial^2 w}{\partial x^2} &= \cos \theta \frac{\partial}{\partial r} \frac{\partial w}{\partial x} - \frac{1}{r} \sin \theta \frac{\partial}{\partial \theta} \frac{\partial w}{\partial x} \\
 &= \frac{\partial^2 w}{\partial r^2} \cos^2 \theta - 2 \frac{\partial^2 w}{\partial r \partial \theta} \frac{\sin \theta \cos \theta}{r} + \frac{\partial w}{\partial r} \frac{\sin^2 \theta}{r} \\
 &\quad + 2 \frac{\partial w}{\partial \theta} \frac{\sin \theta \cos \theta}{r^2} + \frac{\partial^2 w}{\partial \theta^2} \frac{\sin^2 \theta}{r^2}.
 \end{aligned}$$

Similarly,

$$\frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} \sin^2 \theta + 2 \frac{\partial^2 w}{\partial r \partial \theta} \frac{\sin \theta \cos \theta}{r} + \frac{\partial w}{\partial r} \frac{\cos^2 \theta}{r} - 2 \frac{\partial w}{\partial \theta} \frac{\sin \theta \cos \theta}{r^2} + \frac{\partial^2 w}{\partial \theta^2} \frac{\cos^2 \theta}{r^2}$$

and

$$\begin{aligned}
 \frac{\partial^2 w}{\partial x \partial y} &= \frac{\partial^2 w}{\partial r^2} \sin \theta \cos \theta + \frac{\partial^2 w}{\partial r \partial \theta} \frac{\cos 2\theta}{r} - \frac{\partial w}{\partial \theta} \frac{\cos 2\theta}{r^2} \\
 &\quad - \frac{\partial w}{\partial r} \frac{\sin \theta \cos \theta}{r} - \frac{\partial^2 w}{\partial \theta^2} \frac{\sin \theta \cos \theta}{r^2}.
 \end{aligned}$$

Hence, Eq. (3-20) becomes

$$\begin{aligned} \frac{\partial^4 w}{\partial r^4} + \frac{2}{r} \frac{\partial^3 w}{\partial r^3} - \frac{1}{r^2} \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^3} \frac{\partial w}{\partial r} + \frac{2}{r^2} \frac{\partial^4 w}{\partial r^2 \partial \theta^2} \\ - \frac{2}{r^3} \frac{\partial^3 w}{\partial \theta^2 \partial r} + \frac{4}{r^4} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r^4} \frac{\partial^4 w}{\partial \theta^4} = p/D. \end{aligned} \quad (3-21)$$

and the values of M_x , M_y , and M_{xy} become

$$\begin{aligned} M_r &= -D \left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} \right) \\ &= -D \left[\frac{\partial^2 w}{\partial r^2} + \mu \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \right] \end{aligned} \quad (3-22)$$

$$M_t = -D \left(\frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \mu \frac{\partial^2 w}{\partial r^2} \right) \quad (3-23)$$

$$M_{rt} = (1 - \mu)D \left(\frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta} \right). \quad (3-24)$$

The shear forces are expressed as

$$Q_r = -D \frac{\partial}{\partial r} \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \quad (3-25)$$

$$Q_t = -D \frac{1}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right) \quad (3-26)$$

The boundary conditions are:

for simply supported plates

$$w = 0 \quad \text{and} \quad M_r = 0$$

for fixed plates

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial r} = 0$$

for plates with free edge

$$M_r = 0 \quad \text{and} \quad V = \left(Q_r - \frac{1}{r} \frac{\partial M_{rt}}{\partial \theta} \right) = 0.$$

Equation (3-21) is solved by letting

$$w = w_h + w_p.$$

The homogeneous solution, w_h , is expressed (McFarland et al. 1972) by the following Fourier series

$$w_h = \sum_{n=0}^{\infty} f_n(r) \cos n\theta + \sum_{n=1}^{\infty} g_n(r) \sin n\theta. \quad (3-27)$$

Substituting this expression into $\nabla^4 w = 0$ gives

$$\sum_{n=0}^{\infty} \left(\frac{d^4 f_n}{dr^4} + \frac{2}{r} \frac{d^3 f_n}{dr^3} - \frac{1+2n^2}{r^2} \frac{d^2 f_n}{dr^2} + \frac{1+2n^2}{r^3} \frac{df_n}{dr} + \frac{n^2(n^2-4)}{r^4} f_n \right) \cos n\theta \\ + \sum_{n=1}^{\infty} \left(\frac{d^4 g_n}{dr^4} + \frac{2}{r} \frac{d^3 g_n}{dr^3} - \frac{1+2n^2}{r^2} \frac{dg_n}{dr^2} + \frac{1+2n^2}{r^3} \frac{dg_n}{dr} + \frac{n^2(n^2-4)}{r^4} g_n \right) \sin n\theta = 0.$$

This equation is satisfied if

$$\frac{d^4 f_n}{dr^4} + \frac{2}{r} \frac{d^3 f_n}{dr^3} - \frac{1+2n^2}{r^2} \frac{d^2 f_n}{dr^2} + \frac{1+2n^2}{r^3} \frac{df_n}{dr} + \frac{n^2(n^2-4)}{r^4} f_n = 0 \quad (3-28)$$

and

$$\frac{d^4 g_n}{dr^4} + \frac{2}{r} \frac{d^3 g_n}{dr^3} - \frac{1+2n^2}{r^2} \frac{d^2 g_n}{dr^2} + \frac{1+2n^2}{r^3} \frac{dg_n}{dr} + \frac{n^2(n^2-4)}{r^4} g_n = 0. \quad (3-29)$$

Let $f_n(r) = b_n r^m$

and

$g_n(r) = c_n r^m$.

Substituting these equations into Eqs. (3-28) and (3-29) gives

$$m(m-1)(m-2)(m-3) + 2m(m-1)(m-2) - (1+2n^2)m(m-1) + (1+2n^2)m \\ + n^2(n^2-4) = 0.$$

The roots of this equation are

$$m_1 = n, \quad m_2 = -n, \quad m_3 = n+2, \quad m_4 = -n+2.$$

$$\text{If } n = 0, \quad m_1 = m_2 = 0, \quad m_3 = m_4 = 2$$

and

$$f_0(r) = A_0 r^0 + B_0 r^2 + C_0 r^0 \ln r + D_0 r^2 \ln r \\ = A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r.$$

$$\text{If } n = 1, \quad m_1 = m_4 = 1, \quad m_2 = -1, \quad m_3 = 3$$

and

$$f_1 = A_1 r + B_1 r^3 + C_1 r^{-1} + D_1 r \ln r \\ g_1 = E_1 r + F_1 r^3 + G_1 r^{-1} + H_1 r \ln r.$$

Similarly,

$$f_n = A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2} \\ g_n = E_n r^n + F_n r^{-n} + G_n r^{n+2} + H_n r^{-n+2}.$$

Hence, the homogeneous solution, w_h , becomes

$$\begin{aligned} w_h = & A_0 + B_0 r^2 + C_0 \ln r + D_0 r^2 \ln r + (A_1 r + B_1 r^3 + C_1 r^{-1} + D_1 r \ln r) \cos \theta \\ & + (E_1 r + F_1 r^3 + G_1 r^{-1} + H_1 r \ln r) \sin \theta + \sum_{n=2}^{\infty} (A_n r^n + B_n r^{-n} + C_n r^{n+2} \\ & + D_n r^{-n+2}) \cos n\theta + \sum_{n=2}^{\infty} (E_n r^n + F_n r^{-n} + G_n r^{n+2} + H_n r^{-n+2}) \sin n\theta. \end{aligned} \quad (3-30)$$

The particular solution, w_p , is obtained by letting

$$w_p = I_0(r) + \sum_{n=1}^{\infty} [I_n(r) \cos n\theta + J_n(r) \sin n\theta] \quad (3-31)$$

and

$$p = p_0(r) + \sum_{n=1}^{\infty} [p_n(r) \cos n\theta + S_n(r) \sin n\theta] \quad (3-32)$$

where

$$p_n(r) = \frac{1}{\pi} \int_{-\pi}^{\pi} p(r, \theta) \cos n\theta \, d\theta \quad n = 0, 1, 2, \dots$$

$$S_n(r) = \frac{1}{\pi} \int_{-\pi}^{\pi} p(r, \theta) \sin n\theta \, d\theta \quad n = 1, 2, \dots$$

Substituting w_p into the equation $\nabla^4 w = p/D$ gives

$$\begin{aligned} & \frac{d^4 I_0}{dr^4} + \frac{2}{r} \frac{d^3 I_0}{dr^3} - \frac{1}{r^2} \frac{d^2 I_0}{dr^2} + \frac{1}{r^3} \frac{dI_0}{dr} \\ & + \sum_{n=1}^{\infty} \left[\frac{d^4 I_n}{dr^4} + \frac{2}{r} \frac{d^3 I_n}{dr^3} - \frac{1+2n^2}{r^2} \frac{d^2 I_n}{dr^2} + \frac{1+2n^2}{r^3} \frac{dI_n}{dr} + \frac{n^2(n^2-4)}{r^4} I_n \right] \cos n\theta \\ & + \sum_{n=1}^{\infty} \left[\frac{d^4 J_n}{dr^4} + \frac{2}{r} \frac{d^3 J_n}{dr^3} - \frac{1+2n^2}{r^2} \frac{d^2 J_n}{dr^2} + \frac{1+2n^2}{r^3} \frac{dJ_n}{dr} + \frac{n^2(n^2-4)}{r^4} J_n \right] \sin n\theta \\ & = \frac{p_0(r)}{D} + \frac{1}{D} \sum p_n \cos n\theta + \frac{1}{D} \sum S_n \sin n\theta, \end{aligned}$$

from which we obtain the following solution:

$$\frac{d^4 I_0}{dr^4} + \frac{2}{r} \frac{d^3 I_0}{dr^3} - \frac{1}{r^2} \frac{d^2 I_0}{dr^2} + \frac{1}{r^3} \frac{dI_0}{dr} = p_0(r)/D \quad (3-33)$$

$$\frac{d^4 I_n}{dr^4} + \frac{2}{r} \frac{d^3 I_n}{dr^3} - \frac{1+2n^2}{r^2} \frac{d^2 I_n}{dr^2} + \frac{1+2n^2}{r^3} \frac{dI_n}{dr} + \frac{n^2(n^2-4)}{r^4} I_n = p_n/D \quad (3-34)$$

$$\frac{d^4 J_n}{dr^4} + \frac{2}{r} \frac{d^3 J_n}{dr^3} - \frac{1+2n^2}{r^2} \frac{d^2 J_n}{dr^2} + \frac{1+2n^2}{r^3} \frac{dJ_n}{dr} + \frac{n^2(n^2-4)}{r^4} J_n = S_n/D. \quad (3-35)$$

Equation (3-21) is the differential equation for the bending of circular plates and is derived from the expression $\nabla^4 w = p/D$. Its solution is given by Eqs. (3-30) and (3-33) through (3-35).

Example 3-5

Find the bending moment in the plate shown in Fig. 3-12. The load distribution on the plate is given by

$$p = p_o \frac{r}{a} \cos \theta.$$

Solution

Since the applied load is a function of $\cos \theta$, all terms in the homogeneous deflection given by Eq. (3-30) are deleted except

$$w_h = (A_1 r + B_1 r^3 + C_1/r + D_1 r \ln r) \cos \theta. \quad (1)$$

Similarly, Eqs. (3-33) and (3-35) are ignored and Eq. (3-34) is used. The expression for p_n becomes p_1 since the load is a function of θ only. Accordingly,

$$\begin{aligned} p_1(r) &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{p_o r}{a} \cos \theta \right) \cos \theta \, d\theta = \frac{p_o r}{a} \int_{-\pi}^{\pi} \cos^2 \theta \, d\theta \\ &= p_o r / a. \end{aligned}$$

Equation (3-34) becomes

$$\frac{d^4 I_1}{dr^4} + \frac{2}{r} \frac{d^3 I_1}{dr^3} - \frac{3}{r^2} \frac{d^2 I_1}{dr^2} + \frac{3}{r^3} \frac{dI_1}{dr} - \frac{3}{r^4} I_1 = \frac{p_o r}{aD}. \quad (2)$$

Let $I_1 = C_1 r^5 + C_2 r^4 + C_3 r^3 + C_4 r^2 + C_5 r + C_6$.

Substituting this expression into Eq. (2) and solving for the constants C_1 through C_6 gives

$$C_1 = \frac{p_o}{192aD}$$

$$C_2 = C_3 = C_4 = C_5 = C_6 = 0$$

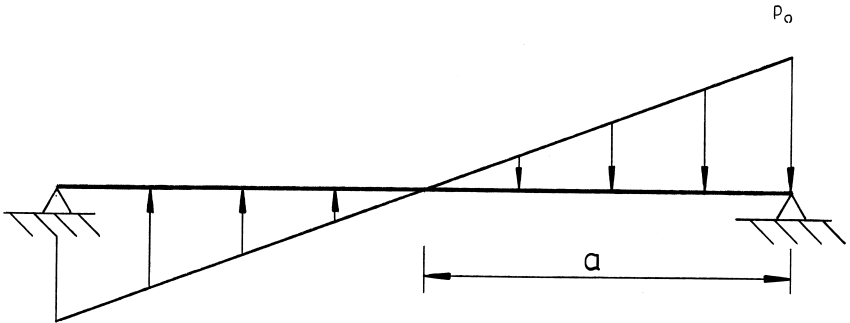


Figure 3-12. Plate with variable load in the θ -direction.

and

$$I_1 = \frac{p_o r^5}{192aD}.$$

The solution for w_p is expressed as

$$w_p = \frac{p_o r^5}{192aD} \cos \theta.$$

combining this expression with Eq. (1) gives the total solution for the deflection.

$$w = \left(A_1 r + B_1 r^3 + C_1/r + D_1 r \ln r + \frac{p_o r^5}{192aD} \right) \cos \theta.$$

Since θ and M_r are finite as $r \rightarrow 0$, constants C_1 and D_1 must be set to zero. The deflection expression then becomes

$$w = \left(A_1 r + B_1 r^3 + \frac{p_o r^5}{192aD} \right) \cos \theta. \quad (3)$$

At $r = a$, $M_r = 0$. Equation (3) gives

$$B_1 = -\frac{2(5 + \mu)}{(3 + \mu)} \frac{p_o a}{192D}.$$

At $r = a$, $w = 0$ and Eq. (3) results in

$$A_1 = \frac{(7 + \mu)}{(3 + \mu)} \frac{p_o a^3}{192D}.$$

The final expression for the deflection can now be written as

$$w = \frac{p_o}{192D} \left(\frac{r^5}{a} - \frac{2(5 + \mu)}{(3 + \mu)} ar^3 + \frac{(7 + \mu)}{(3 + \mu)} a^3 r \right) \cos \theta.$$

The equations for M_r and M_t can now be obtained and are expressed as

$$M_r = \frac{p_o a^2 (5 + \mu)}{48} (r/a) \left(1 - \frac{r^2}{a^2} \right) \cos \theta$$

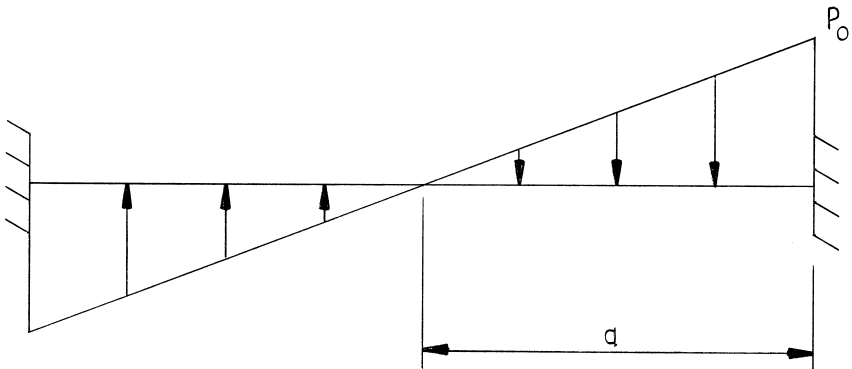
$$M_t = \frac{4p_o a^2}{192} \left[\left(\frac{r}{a} \right) \frac{(5 + \mu)(1 + 3\mu)}{(3 + \mu)} - \frac{r^3}{a^3} (1 + 5\mu) \right] \cos \theta.$$

Problems

3-11 Find the maximum bending moment in the plate. The edge of the plate is fixed and the applied load is expressed as

$$p = \frac{p_o r}{a} \cos \theta.$$

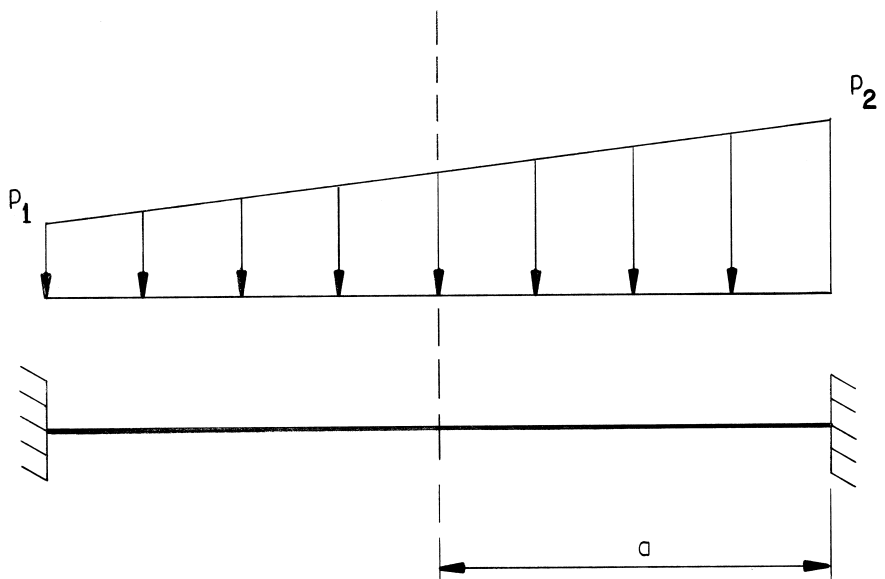
3-12 Find the expression for the bending moments in the plate shown.



Prob. 3-11. Fixed plate with variable load.

3-4 Plates on an Elastic Foundation

Power and petrochemical plants as well as refineries use evaporators, condensers, and heat exchange units as part of their daily operations. These units consist of two perforated circular plates, called tubesheets, that are braced by a number of tubes as shown in Fig. 3-13. The tubesheets and tubes are inserted in a vessel consisting of a cylindrical shell and two end closures. Fluid passing around the



Prob. 3-12. Fixed plate with hydrostatic load.

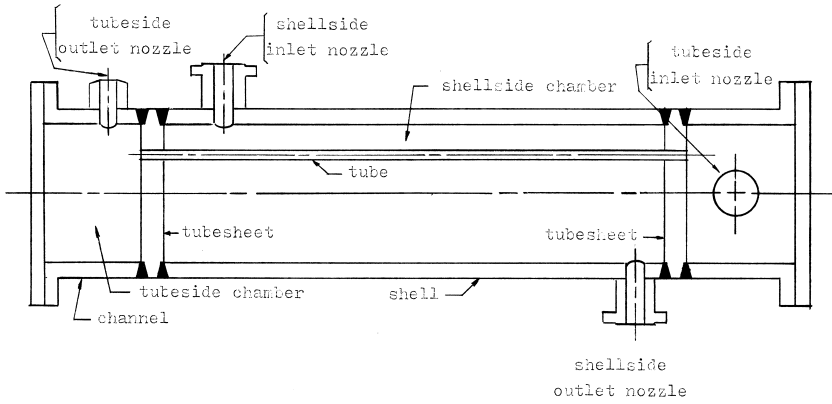


Figure 3-13. Heat exchanger.

outside surface of the tubes exchanges heat with a different fluid passing through the tubes. The tubesheets are assumed to be supported by both the shell and tubes and are analyzed as circular plates on an elastic foundation. Referring to Figs. 3-2 and 3-14, it is seen that the foundation pressure f acts opposite the applied pressure p . Hence Eq. (3-9) can be expressed as (Hetenyi 1964)

$$\frac{1}{r} \frac{d}{dr} \left\{ r \frac{r}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dw}{dr} \right) \right] \right\} = \frac{p_r - f_r}{D} \quad (3-36)$$

or

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) w = \frac{p_r - k_w}{D} \quad (3-37)$$

where

f_r = load exerted by the elastic foundation;

$f_r = k_o w$;

k_o = foundation modulus defined as the modulus of elasticity of foundation divided by the depth of foundation, psi per inch.

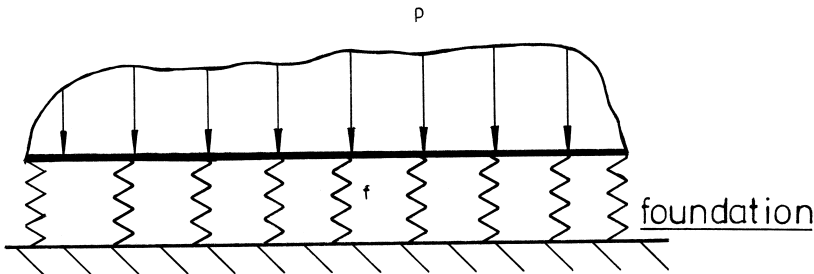


Figure 3-14. Plate on elastic foundation.

A plate on an elastic foundation that is subjected to uniform pressure will settle uniformly without developing any bending moments. If a support is placed at the edge of such a plate, then bending moments and shear are developed because of the nonuniform settlement caused by the boundary condition.

Accordingly, we can investigate the effects of the various boundary conditions on the plate stress by allowing the applied pressure to be set to zero. Letting

$$\alpha = \sqrt[4]{k_0/D}$$

the differential equation becomes

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}\right) \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}\right) w + \alpha^4 w = 0.$$

Let $\alpha r = \sqrt[4]{-1} \rho$.

Then the differential equation becomes

$$\nabla^4 w - w = 0 \quad (3-38)$$

where

$$\nabla^4 = \left(\frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho}\right)^2.$$

Equation (3-38) can be written either as

$$\nabla^2(\nabla^2 w + w) - (\nabla^2 w + w) = 0$$

or

$$\nabla^2(\nabla^2 w - w) + (\nabla^2 w - w) = 0.$$

The solution is a combination of

$$\nabla^2 w + w = 0$$

and

$$\nabla^2 w - w = 0.$$

The first equation can be written as

$$\frac{d^2 w}{d\rho^2} + \frac{1}{\rho} \frac{dw}{d\rho} + w = 0$$

and the solution is expressed in terms of Bessel function, see Appendix B, as

$$w = A_1 J_0(\rho) + A_2 Y_0(\rho).$$

The second equation has a solution in the form of

$$w = A_3 J_0(i\rho) + A_4 Y_0(i\rho).$$

Hence, the total solution is written as

$$w = A_1 J_0(\pm \alpha r \sqrt{i}) + A_2 Y_0(\pm \alpha r \sqrt{i}) + A_3 J_0(\pm \alpha r \sqrt{-i}) + A_4 Y_0(\pm \alpha r \sqrt{-i}).$$

This equation can be written as (Hetenyi 1964)

$$w = C_1 Z_1(\alpha r) + C_2 Z_2(\alpha r) + C_3 Z_3(\alpha r) + C_4 Z_4(\alpha r) \quad (3-39)$$

where the functions Z_1 to Z_4 are modified Bessel functions given in Appendix B.

Example 3-6

A tubesheet in a heat exchanger Fig. 3-15 is subjected to edge load, Q_o , caused by the difference in expansion between the supporting tubes and the cylindrical shell. Find the expression for the deflection of the tubesheet due to force Q_o if the tubesheet is assumed simply supported at the edges.

Solution

From Eq. (3-39),

$$w = C_1 Z_1(\alpha r) + C_2 Z_2(\alpha r) + C_3 Z_3(\alpha r) + C_4 Z_4(\alpha r). \quad (1)$$

The first constant is determined from the boundary condition at $r = 0$, where the slope dw/dr is equal to zero due to symmetry. Hence,

$$\left. \frac{dw}{dr} \right|_{r=0} = C_1 \alpha Z_1'(\alpha r) + C_2 \alpha Z_2'(\alpha r) + C_3 \alpha Z_3'(\alpha r) + C_4 \alpha Z_4'(\alpha r).$$

From Fig. B-3 of Appendix B, the quantity $Z_4'(0)$ approaches infinity as r approaches zero. Hence, C_4 must be set to zero. The second constant is determined from the boundary at $r = 0$ where the shearing force, Q , is zero due to symmetry. The shearing force is expressed as

$$Q = D \left(\frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} \right). \quad (2)$$

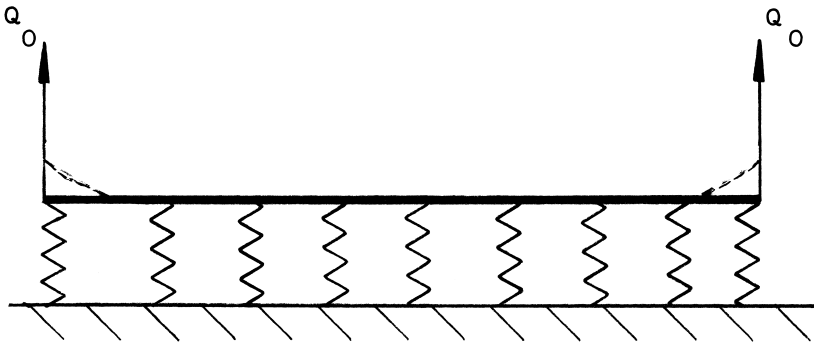


Figure 3-15. Plate on elastic foundation with edge load.

The derivatives of the first term in Eq. (1) are

$$\frac{dw}{dr} = C_1 \alpha Z_1'(\alpha r)$$

$$\frac{d^2w}{dr^2} = C_1 \alpha^2 Z_1''(\alpha r)$$

or, from Appendix B,

$$\frac{d^2w}{dr^2} = C_1 \left[\alpha^2 Z_2(\alpha r) - \frac{\alpha}{r} Z_1'(\alpha r) \right].$$

The third derivative is

$$\frac{d^3w}{dr^3} = C_1 \left[\alpha^3 Z_2'(\alpha r) - \frac{\alpha^2}{r} Z_1''(\alpha r) + \frac{\alpha}{r^2} Z_1'(\alpha r) \right]$$

or, from Appendix B,

$$\frac{d^3w}{dr^3} = C_1 \left\{ \alpha^3 Z_2'(\alpha r) - \frac{1}{r} \left[\alpha^2 Z_2(\alpha r) - \frac{\alpha}{r} Z_1'(\alpha r) \right] + \frac{\alpha}{r^2} Z_1'(\alpha r) \right\}.$$

Substituting these expressions into Eq. (2) yields

$$Q = D[C_1 \alpha^3 Z_2'(\alpha r)].$$

The derivatives of Z_2 and Z_3 in Eq. (1) are similar to those for Z_1 . Thus, the total expression for Q in Eq. (2) becomes

$$Q = D[C_1 \alpha^3 Z_2'(\alpha r) - C_2 \alpha^3 Z_1'(\alpha r) + C_3 \alpha^3 Z_4'(\alpha r)].$$

At $r = 0$, $Q = 0$ due to symmetry. From Fig. B-3 of Appendix B, Z_1' and Z_2' have a finite value at $r = 0$ while Z_4' approaches ∞ . Hence, C_3 must be set to zero. At the boundary condition $r = a$ we have

$$M_r|_{r=a} = 0 = -D \left(\frac{d^2w}{dr^2} + \frac{\mu}{r} \frac{dw}{dr} \right) \quad (3)$$

and

$$Q|_{r=a} = Q_0 = D \left(\frac{d^3w}{dr^3} + \frac{1}{r} \frac{d^2w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} \right). \quad (4)$$

Substituting Eq. (1) into Eq. (3) yields

$$C_2 = C_1 \frac{\alpha Z_2(\alpha a) - \frac{1-\mu}{a} Z_1'(\alpha a)}{\alpha Z_1(\alpha a) + \frac{1-\mu}{a} Z_2'(\alpha a)}. \quad (5)$$

And from Eqs. (4) and (5) we get

$$C_1 = \frac{-Q_0 \alpha}{k_0 F} \left[Z_1(\alpha a) + \frac{1-\mu}{\alpha a} Z_2'(\alpha a) \right]$$

where

$$F = Z_1(\alpha\alpha)Z_2'(\alpha\alpha) - Z_1'(\alpha\alpha)Z_2(\alpha\alpha) + \frac{1-\mu}{\alpha\alpha}[Z_1'^2(\alpha\alpha) + Z_2'^2(\alpha\alpha)].$$

Substituting the expression for C_1 into Eq. (5) gives

$$C_2 = \frac{-Q_0\alpha}{k_0F} \left[Z_2(\alpha\alpha) - \frac{1-\mu}{\alpha\alpha} Z_1'(\alpha\alpha) \right].$$

With C_1 and C_2 known, and $C_3 = C_4 = 0$, Eq. (1) can be solved for moments and shears throughout the plate.

Problems

3-13 Show that the maximum deflection of a circular plate on an elastic foundation subjected to a concentrated load, F , in the center is given by the following expression when the radius of the plate is assumed infinitely large

$$w_{\max} = \frac{F}{8D\alpha^2}.$$

3-14 What are the values of C_1 and C_2 in Example 3-6 if the shear force Q_0 is replaced by a bending moment M_0 ?

3-5 Plates with Variable Boundary Conditions

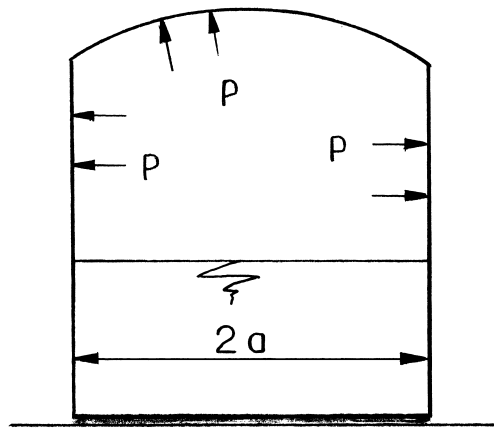
In many structures such as large oil storage tanks, Fig. 3-16a, the surface pressure above the contents causes an uplift force in the cylindrical shell. This force is normally transferred to the foundation through the anchor bolts. Many tanks, however, are not anchored to the foundation, especially in earthquake zones, to avoid damage to the tanks and their attachments. In such cases the uplift force due to surface pressure and earthquake loads is transferred to the base plate as shown in Fig. 3-16b. The edge of the plate tends to lift up and the rest of the plate is kept in place by the pressure of the tank contents. The deflection of such a plate is obtained from Eqs. (3-30) and (3-33) as

$$w = A + Br^2 + C \ln r + Fr^2 \ln r + \frac{pr^4}{64D} \quad (3-40)$$

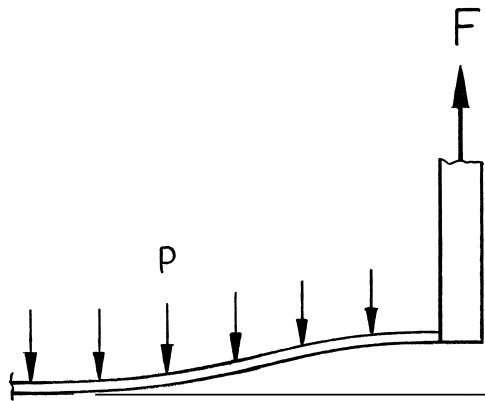
where the constants A , B , C , and F are determined from the boundary conditions.

At $r = a$, two boundary conditions can be specified. The first is the uplift force Q_0 . The other boundary condition is obtained by specifying either M_r or θ . The other boundary conditions are obtained from Fig. 3-17 by assuming an unknown dimension $r = b$ at which the following boundary conditions are satisfied.

$$w = 0, \quad dw/dr = 0, \quad M_r = 0.$$



(a)



(b)

Figure 3-16. Uplift force in a flat bottom tank.

These three boundary conditions plus the two at $r = a$ are used to solve the unknowns b , A , B , C , and F . As the constants A , B , C , and F are functions of b , the five equations obtained from the boundary conditions cannot be solved directly. A practical solution, however, can easily be obtained by writing

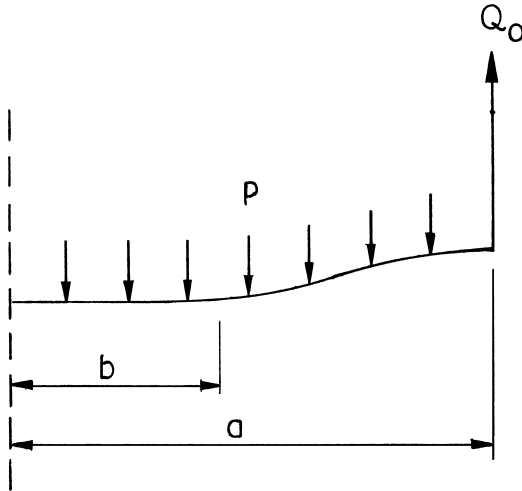


Figure 3-17. Uplift force Q .

a small computer program that increments various values of b until a solution is obtained.

Example 3-7

The tank shown in Fig. 3-18 is subjected to an earthquake motion that results in an upward force at the cylinder-to-plate junction of 234 lbs/inch. Determine the maximum stress in the bottom plate and the maximum uplift. Assume the shell-to-plate junction to have zero rotation due to the cylindrical shell being substantially thicker than the base plate. Let $\mu = 0.3$ and $E = 29,000$ ksi.

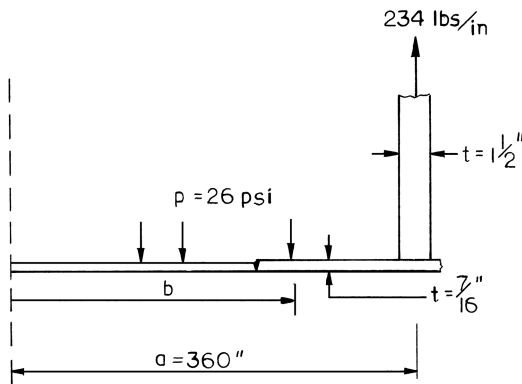


Figure 3-18. An API flat bottom tank.

Solution

At $r = b$ the deflection, w , is zero and Eq (3-40) becomes

$$A + Bb^2 + C \ln b + Fb^2 \ln b = \frac{-pb^4}{64D}. \quad (1)$$

At $r = b$ the slope $dw/dr = 0$. Equation (3-40) gives

$$2Bb + C/b + Fb(2 \ln b + 1) = \frac{-4pb^3}{16D}. \quad (2)$$

At $r = b$, $M_r = 0$ and

$$M_r = -D \left(\frac{d^2 w}{dr^2} + \frac{\mu}{r} \frac{dw}{dr} \right) = 0.$$

Substituting w into this equation gives

$$3.7143Bb^2 - C - Fb^2(3.7143 \ln b + 4.7143) + \frac{4.1743pb^4}{16D} = 0. \quad (3)$$

at $r = a$, $dw/dr = 0$ and

$$2Ba + C/a + Fa(2 \ln a + 1) = \frac{-4pa^3}{64D}. \quad (4)$$

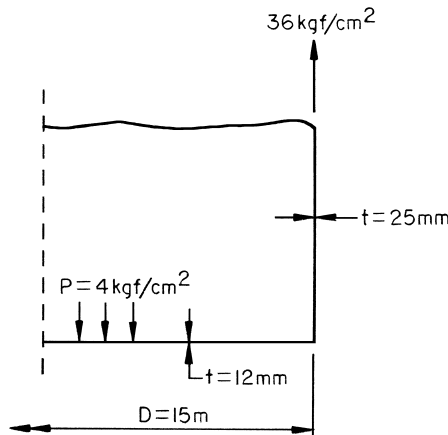
At $r = a$, $Q = Q_o$ and

$$Q_o = D \left(\frac{d^3 w}{dr^3} + \frac{1}{r} \frac{d^2 w}{dr^2} - \frac{1}{r^2} \frac{dw}{dr} \right),$$

which gives

$$F = \frac{Q_o a}{4D} - \frac{pa^2}{8D}. \quad (5)$$

Combining Eqs. (2), (3), (4), and (5) results in an equation that relates the unknown quantity b to the known loads Q_o and p . By placing all terms on one



Prob 3-15. Forces on a flat bottom tank.

Table 3-1. Circular plates of uniform thickness

Case Number	Maximum Values
1.	$S_r = S_t = \frac{3pa^2}{8t^2}(3 + \mu) \text{ at center}$ $w = \frac{3pa^4(1 - \mu)(5 + \mu)}{16Et^3} \text{ at center}$ $\theta = -\frac{3pa^3(1 - \mu)}{2Et^3} \text{ at edge}$ For $\mu = 0.3$, $S_r = S_t = \frac{1.238pa^2}{t^2} \text{ at center}$ $w = \frac{0.696pa^4}{Et^3} \text{ at center}$ $\theta = -\frac{1.050pa^3}{Et^3} \text{ at edge}$
2.	$S_r = -\frac{3pa^2}{4t^2} \text{ at edge}$ $w = -\frac{3pa^4(1 - \mu^2)}{16Et^3} \text{ at center}$ For $\mu = 0.3$ $S_r = -\frac{3pa^2}{4t^2} \text{ at edge}$ $w = \frac{0.171pa^4}{Et^3} \text{ at center}$
3.	$S_r = S_t = 6M_o/t^2 \text{ at any point}$ $w = \frac{6M_o a^2(1 - \mu)}{Et^3} \text{ at center}$ $\theta = -\frac{12M_o a(1 - \mu)}{Et^3} \text{ at edge}$ For $\mu = 0.3$ $S_r = S_t = 6M_o/t^2 \text{ at any point}$ $w = \frac{4.20M_o a^2}{Et^3} \text{ at center}$ $\theta = -\frac{8.40M_o a}{Et^3} \text{ at edge}$

Notation: a = outside radius of plate; b = radius of applied load; E = modulus of elasticity; p = applied load; r = radius; S_t = tangential stress; S_r = radial stress; t = thickness; w = deflection; θ = rotation, radians; μ = Poisson's ratio.

side of the equation, a computer program can be written to increment the quantity b until a solution is found that satisfies this equation. Using such a program results in a value of b of 346.2 inches. With this dimension known, the constants A , B , C , and F can now be determined and are given by

$$\begin{aligned} A &= 1.1347 \times 10^{11} & B &= 2.5450 \times 10^6 \\ C &= -2.4615 \times 10^{10} & F &= -4.0014 \times 10^5. \end{aligned}$$

The maximum moment, M_r , is found to be 787.9 inch-lbs/inch and the maximum deflection at the edge as 0.02 inch.

$$\begin{aligned} \text{maximum stress in bottom plate} &= 6M/t^2 = 6 \times 787.9/0.4375^2 \\ &= 24,700 \text{ psi} \end{aligned}$$

Problem

3-15 Find the stress in the bottom floor plate of the oil tank shown. Let $E = 200,000$ MPa and $\mu = 0.30$.

3-6 Design of Circular Plates

The references cited in Sections 1-9 and 2-5 for rectangular plates also contain numerous tables for calculating maximum stress and deflection in circular plates of uniform thickness subjected to various loading and boundary conditions. For concrete slabs, extra precaution must be given to placement of reinforcing bars as discussed in Example 3-1.

Circular plates are used as end closures in many shell structures such as reactors, heat exchangers, and distillation towers. Discussion of the interaction of circular plates with various shells will be discussed in later chapters. Table 3-1 lists a few loading conditions that will be utilized later when the interaction of plate and shell components is considered.

An approximate deflection and stress in perforated circular plates are obtained from the theoretical analysis of solid plates modified to take into consideration the effect of the perforations. One procedure that is commonly used is given in the ASME Code, Section VIII-2, Appendix 4. The code uses equivalent values of Poisson's ratio and modulus of elasticity in the theoretical equations for the deflection and stress of solid plates to obtain approximate values for perforated plates. The equivalent values are functions of the pitch and diameter of the perforations. The procedure is based, in part, on O'Donnell's work (O'Donnell and Langer, 1962).

The design of heat exchangers is based on Eq. (3-39) and its solution as shown in Example 3-6. Many codes and standards such as ASME-VIII and TEMA (TEMA 1999) simplify the solution to a set of curves and equations suitable for design purposes.

Plates of Various Shapes and Properties

4-1 Introduction

Many structures (Fig. 4-1) consist of plates with shapes other than rectangular or circular. In this chapter a brief discussion of elliptic and triangular plates is given. The solution of other shapes is obtained by approximate solutions similar to those discussed in Chapter 5.

Many structural components such as bridge decks, reinforced concrete slabs, corrugated sheet plates, and composite materials (Fig. 4-1) have physical properties that are different in the x - and y -axes. Accordingly, the equations derived in Chapter 1 cannot be used directly to analyze these components. Rather, a modified theory is needed and is referred to as the orthotropic plate theory. A brief discussion of this theory is given at the end of this chapter.

4-2 Elliptic Plates

The shape of an elliptic plate, Fig. 4-2, is expressed by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (4-1)$$

where a and b are the major and minor axes. The boundary conditions for an elliptic plate with a fixed boundary, Fig. 4-2, are given by

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial n} = 0$$

where n is normal to the plate edge.

An expression for the deflection of a uniformly loaded plate that satisfies the boundary conditions is given by

$$w = K \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)^2. \quad (4-2)$$



Figure 4-1. F117 Fighter (Photo by: Lockheed Martin Corporation. Photography by: Denny Lombard and Eric Schulzinger.)

Substituting Eq. (4-2) into the differential Eq. (1-26) gives

$$K = \frac{p}{8D} \frac{a^4 b^4}{3a^4 + 3b^4 + 2a^2 b^2} \quad (4-3)$$

and the moments are obtained from Eq. (1-17) as

$$M_x = -4DK \left[x^2 \left(\frac{3}{a^4} + \frac{\mu}{a^2 b^2} \right) + y^2 \left(\frac{1}{a^2 b^2} + \frac{3\mu}{b^4} \right) - \left(\frac{1}{a^2} + \frac{\mu}{b^2} \right) \right] \quad (4-4)$$

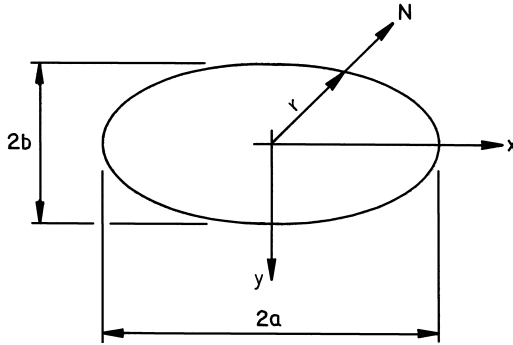


Figure 4-2. Elliptical plate.

$$M_y = -4DK \left[x^2 \left(\frac{3\mu}{a^4} + \frac{1}{a^2b^2} \right) + y^2 \left(\frac{\mu}{a^2b^2} + \frac{3}{b^4} \right) - \left(\frac{\mu}{a^2} + \frac{1}{b^2} \right) \right] \quad (4-5)$$

$$M_{xy} = -8DK(1 - \mu) \frac{xy}{a^2b^2}. \quad (4-6)$$

For simply supported plates, the expression for deflection is more complicated than that given in Eq. (4-2). The solution has been developed by many authors such as Perry (Perry 1950).

Example 4-1

Determine the required thickness of a fixed elliptic plate with $a = 20$ inches and $b = 15$ inches due to a pressure of 100 psi. Let $\mu = 0.3$ and the allowable stress equal 15,000 psi.

Solution

From Eq. (4-3), $K = 124,711/D$, and Eqs. (4-4), (4-5), and (4-6) give

$$M_x = -498,844 \left(\frac{x^2}{45,283} + \frac{y^2}{34,615} - \frac{1}{261} \right)$$

$$M_y = -498,844 \left(\frac{x^2}{59,751} + \frac{y^2}{15,976} - \frac{1}{192.5} \right)$$

$$M_{xy} = -7.76xy.$$

At $x = 20$ inches and $y = 0$ inches, maximum value of $M_x = -2495$ inch-lbs/inch.

At $x = 0$ inches and $y = 15$ inches, maximum value of $M_y = -4434$ inch-lbs/inch.

At $x = 14.02$ inches and $y = 10.7$ inches, maximum value of $M_{xy} = 1164$ inch-lbs/inch.

$$t = \sqrt{\frac{6 \times 4434}{15,000}}$$

$$= 1.33 \text{ inches.}$$

4-3 Triangular Plates

The solution of a uniformly loaded simply supported isosceles right triangular plate, ABO , of length a is obtained from Fig. 4-3. The plate is assumed to be loaded over a small area of dimension $c \times d$ with a downward load p . In order to find a solution that satisfies the simply supported boundary along line AB , we need to apply a mirror image of the load at a point on the other side of the boundary on a fictitious extension of the plate, ABC , as shown in Fig. 4-3. This fictitious load, p' , has an upwards direction. The bending couple due to p' and p results in a zero bending moment and deflection along boundary AB . The deflection due to p is obtained from Example 1-3 with $a = b$ and $c = d$. This gives

$$w_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} \quad (4-7)$$

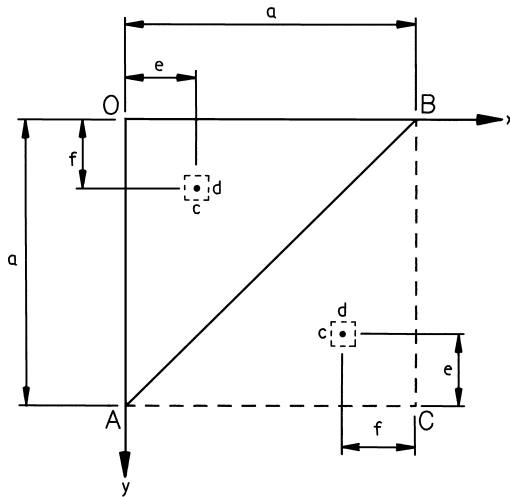


Figure 4-3. Triangular plate.

where

$$w_{mn} = \frac{16p}{\pi^6 Dc^2} \frac{\sin \frac{m\pi e}{a} \sin \frac{m\pi c}{2a} \sin \frac{n\pi f}{a} \sin \frac{n\pi c}{2a}}{mn[(m/a)^2 + (n/a)^2]^2}. \quad (4-8)$$

For load p' we substitute in Example 1-3 the quantity $(a - f)$ for e and the quantity $(a - e)$ for f . This gives

$$w_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} g_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} \quad (4-9)$$

where

$$g_{mn} = \frac{16p'}{\pi^6 Dc^2} \frac{\sin \frac{m\pi f}{a} \sin \frac{m\pi c}{2a} \sin \frac{n\pi e}{a} \sin \frac{n\pi c}{2a} \cos m\pi \cos n\pi}{mn[(m/a)^2 + (n/a)^2]^2} \quad (4-10)$$

and the total solution of the triangular plate is given by

$$w = w_1 + w_2. \quad (4-11)$$

The deflection of a simply supported equilateral triangular plate of length a is obtained by defining the coordinate system as shown in Fig. 4-4. The equations for the boundary conditions become

$$\begin{aligned} x &= -\frac{\sqrt{3}}{6}a \quad \text{at boundary } AB \\ y &= -\frac{1}{\sqrt{3}}x + \frac{a}{3} \quad \text{at boundary } BC \\ y &= -\frac{1}{\sqrt{3}}x - \frac{a}{3} \quad \text{at boundary } AC. \end{aligned}$$

Hence, a deflection expression that vanishes at the boundaries can be expressed by

$$w = K \left(x + \frac{\sqrt{3}}{6}a \right) \left(\frac{x}{\sqrt{3}} + y - \frac{a}{3} \right) \left(\frac{x}{\sqrt{3}} - y - \frac{a}{3} \right)$$

or

$$w = \frac{K}{6} \left(2x^3 - \sqrt{3}x^2a - 6xy^2 - \sqrt{3}ay^2 + \frac{\sqrt{3}}{9}a^3 \right). \quad (4-12)$$

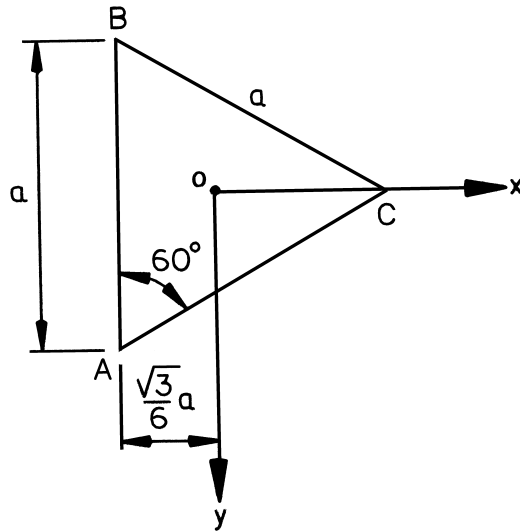


Figure 4-4. Equilateral plate.

Woinowsky-Kreiger (Timoshenko and Woinowsky-Kreiger 1959) obtained a value of K that satisfies the simply supported boundary condition in the form of

$$K = \frac{P}{64aD} (a^2 - 3x^2 - 3y^2). \quad (4-13)$$

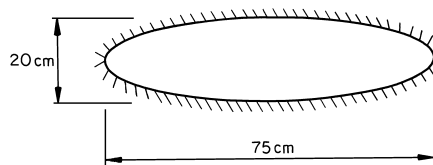
Problems

4-1 What is the required thickness of the fixed elliptic plate due to a 7 kgf/cm^2 pressure? Let the allowable stress be 1400 kgf/cm^2 and $\mu = 0.26$.

4-2 Determine the expressions for M_x , M_y , and M_{xy} for a simply supported isosceles right triangular plate.

4-3 Determine the expressions for M_x , M_y , and M_{xy} for a simply supported equilateral triangular plate.

4-4 Compare the maximum moment value obtained from Problem 4-3 with that obtained from a circumscribed circle.



Prob. 4-1. Elliptical plate with fixed edge.

4-4 Orthotropic Plate Theory

In the discussion of plates so far, the material was assumed homogeneous and isotropic. In an isotropic material subjected to an axial stress in a principal direction, the major deformation occurs in the direction of applied load. Lateral deformation of smaller magnitude (Fig. 4-5a) occurs in the other principal directions. Also, shearing stress causes only shearing deformation as discussed in Section 1-3. The deformation is dependent on the elastic constants E and μ . Many materials of construction such as steel, aluminum, and titanium fall into this category.

In orthotropic materials stressed in one of the principal directions, the lateral deformation in the other principal directions could be smaller or larger than the

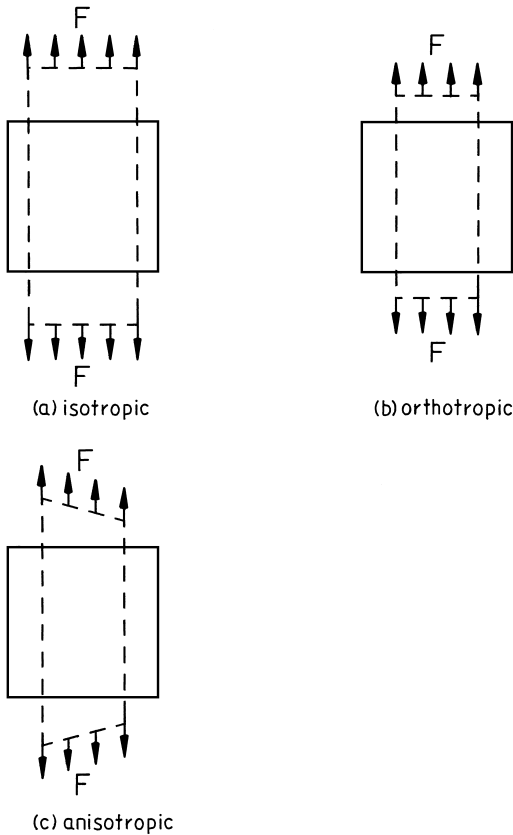


Figure 4-5. Deformation of a stressed element.

deformation in the direction of the applied stress depending on the material properties (Fig. 4-5b). Also, the magnitude of the shearing deformation (Jones 1975) is independent of the elastic constants. Some materials of construction that fall into this category are bridge steel decks (Troitsky 1987), reinforced concrete, plywood sheets, and composite materials (ASMEc 2001).

In anisotropic plates, or orthogonal plates stressed in other than the principal axes, the applied stress in a given direction causes not only extension in the same direction and deformation in the other two directions, but also shearing deformation (Fig. 4-5c). Similarly shearing stress causes not only shearing deformations, but also axial deformations. The state of stress in an anisotropic plate is very complicated and is beyond the scope of this book.

The development of the plate theory for orthotropic materials is similar to that of isotropic materials. The stress-strain Eq. (1-14) can be rewritten as (McFarland et al. 1972)

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} E_1 & E_{12} & 0 \\ E_{12} & E_2 & 0 \\ 0 & 0 & G \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \quad (4-14)$$

where E_1 , E_2 , E_{12} , and G are constants defined as

$$E_1 = \frac{E_x}{1 - \mu_x \mu_y}$$

$$E_2 = \frac{E_y}{1 - \mu_x \mu_y}$$

$$E_{12} = \mu_y E_1 = \mu_x E_2$$

$$G = \frac{\tau_{xy}}{\gamma_{xy}}$$

where

E_x = modulus of elasticity in the x -direction;

E_y = modulus of elasticity in the y -direction;

μ_x = contraction in the y -direction due to stress in the x -direction;

μ_y = contraction in the x -direction due to stress in the y -direction;

τ_{xy} = shearing stress in the x - y direction;

γ_{xy} = shearing strain in the x - y direction.

The strain deflection expressions given by Eq. (1-12) are also valid for orthotropic plates as they are a function of geometry only. Substituting Eq. (1-12)

into Eq. (4-14) gives

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = -z \begin{bmatrix} E_1 & E_{12} & 0 \\ E_{12} & E_2 & 0 \\ 0 & 0 & 2G \end{bmatrix} \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix}. \quad (4-15)$$

The moment expressions given by Eq. (1-17) become

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = - \begin{bmatrix} D_x & D_{xy} & 0 \\ D_{xy} & D_y & 0 \\ 0 & 0 & 2D_s \end{bmatrix} \begin{bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 w}{\partial x \partial y} \end{bmatrix} \quad (4-16)$$

where D_x , D_y , D_{xy} , and D_s are bending stiffness constants defined as

$$\begin{aligned} D_x &= \frac{E_1 t^3}{12} & D_y &= \frac{E_2 t^3}{12} \\ D_{xy} &= \frac{E_{12} t^3}{12} & D_s &= \frac{G t^3}{12}. \end{aligned} \quad (4-17)$$

The shear expressions Eqs. (1-27) and (1-28) become

$$Q_x = -\frac{\partial}{\partial x} \left(D_x \frac{\partial^2 w}{\partial x^2} + H \frac{\partial^2 w}{\partial y^2} \right) \quad (4-18)$$

$$Q_y = -\frac{\partial}{\partial y} \left(H \frac{\partial^2 w}{\partial x^2} + D_y \frac{\partial^2 w}{\partial y^2} \right) \quad (4-19)$$

where

$$H = D_{xy} + 2D_s. \quad (4-20)$$

The plate differential Eq. (1-25) becomes

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = p(x, y). \quad (4-21)$$

Equation (4-21) is referred to as the Huber's differential equation of orthotropic plates (Troitsky 1987).

For simply supported plates, the expressions for deflection and load are given by Eqs. (1-41) and (1-39). For a uniformly loaded plate, Eq. (1-40) gives

$$p_{mn} = \frac{16p_0}{\pi^2 mn}$$

where m and n are odd.

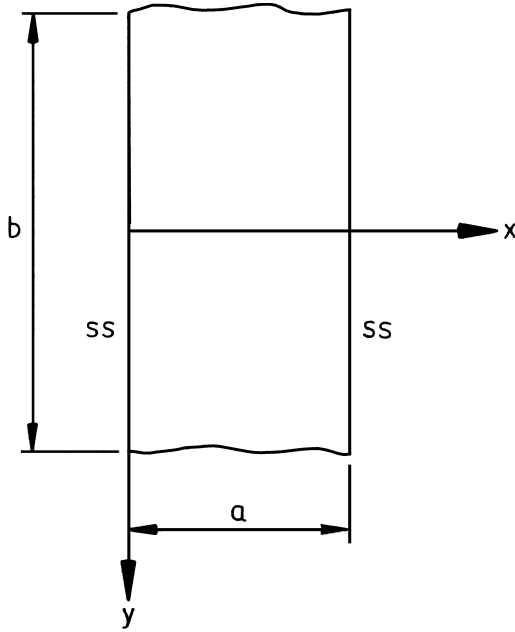


Figure 4-6. Orthotropic plate.

Substituting this expression and Eqs. (1-39) and (1-41) into Eq. (4-21) gives

$$w_{mn} = \frac{16p_0}{\pi^6} \frac{1}{mn \left(\frac{m^4}{a^4} D_x + \frac{2m^2 n^2}{a^2 b^2} H + \frac{n^4}{b^4} D_y \right)}. \quad (4-22)$$

The deflection is obtained by substituting this expression into Eq. (1-41).

For isotropic plates, $D_x = D_y = H = D$. Thus, Eq. (4-22) reduces to the value of w_{mn} given in Example 1-2 for isotropic plates.

For a plate simply supported along two sides and infinitely long along the other two sides (Fig. 4-6), we define the deflection as

$$w_h = \sum_{m=1}^{\infty} f_m(y) \sin \frac{m\pi x}{a}.$$

Then the homogeneous solution of Eq. (4-21) becomes

$$D_x \left(\frac{m\pi}{a} \right)^4 f_m - 2H \left(\frac{m\pi}{a} \right)^2 \frac{d^2 f_m}{dy^2} + D_y \frac{d^4 f_m}{dy^4} = 0.$$

Let $f_m = C e^{gy}$.

Then,

$$D_x \left(\frac{m\pi}{a} \right)^4 - 2Hg^2 \left(\frac{m\pi}{a} \right)^2 + D_y g^4 = 0$$

and

$$\left. \begin{aligned} g_1 &= \pm \frac{m\pi}{a} \sqrt{\frac{1}{D_y} \left(H + \sqrt{H^2 - D_x D_y} \right)} \\ g_3 &= \pm \frac{m\pi}{a} \sqrt{\frac{1}{D_y} \left(H - \sqrt{H^2 - D_x D_y} \right)}, \end{aligned} \right] \quad (4-23)$$

and the homogeneous solution becomes

$$w_h = \sum_{m=1}^{\infty} (A_m e^{g_1 y} + B_m e^{g_2 y} + C_m e^{g_3 y} + D_m e^{g_4 y}) \sin \frac{m\pi x}{a}. \quad (4-24)$$

The particular solution depends on the applied loads. If we assume a uniformly loaded plate, then

$$p(x) = \sum_{m=1}^{\infty} p_m \sin \frac{m\pi x}{a}$$

where

$$\begin{aligned} p_m &= \frac{2}{a} \int_0^a p(x) \sin \frac{m\pi x}{a} dx \\ p_m &= \frac{4p_0}{m\pi} \quad m = 1, 3, \dots \end{aligned}$$

Substituting this expression into Eq. (4-21) results in

$$D_x \left(\frac{m\pi}{a} \right)^4 a_m - 2H \left(\frac{m\pi}{a} \right)^2 \frac{d^2 a_m}{dy^2} + D_y \frac{d^4 a_m}{dy^4} = p_m.$$

The particular solution is

$$a_m = \frac{p_m}{D_x} \left(\frac{a}{m\pi} \right)^4$$

and

$$w_p = \sum_{m=1}^{\infty} \frac{p_m}{D_x} \left(\frac{a}{m\pi} \right)^4 \sin \frac{m\pi x}{a}. \quad (4-25)$$

The total solution becomes

$$w = \sum_{m=1}^{\infty} \left[A_m e^{g_1 y} + B_m e^{g_2 y} + C_m e^{g_3 y} + D_m e^{g_4 y} + \frac{p_m}{D_x} \left(\frac{a}{m\pi} \right)^4 \right] \sin \frac{m\pi x}{a}. \quad (4-26)$$

As y gets larger, the quantities $e^{g_{1y}}$ and $e^{g_{3y}}$ tend to approach infinity. Thus, A_m and C_m are set to zero and Eq. (4-26) reduces to

$$w = \sum_{m=1}^{\infty} \left[B_m e^{g_{2y}} + D_m e^{g_{4y}} + \frac{p_m}{D_x} \left(\frac{a}{m\pi} \right)^4 \right] \sin \frac{m\pi x}{a}. \quad (4-27)$$

Problems

4-5 Find the maximum stress in a simply supported rectangular plate due to a uniform pressure of 8 psi. The plate is constructed of a boron/epoxy laminate with $E_x = 30,000$ ksi, $E_y = 3000$ ksi, $G = 1000$ ksi, $\mu_x = 0.30$, and $\mu_y = 0.03$. Let $a = 30$ inches, $b = 25$ inches, and $t = 1/4$ inch. Compare the result with that of an equivalent isotropic plate of $E = 16,500$ ksi and $\mu = 0.165$.

4-6 Use Eq. (4-27) to solve Problem 4-5. Let $a = 30$ inches and $b = 100$ inches in Fig. 4-6.

4-5 Orthotropic Materials and Structural Components

Solution of Eq. (4-25) requires specific values of the bending stiffness constants D_x , D_y , D_{xy} , and D_s . These constants must be determined experimentally or empirically for various materials and structural components. Many references are available for orthotropic plate design such as McFarland (McFarland et al. 1972), Timoshenko (Timoshenko and Woinowsky-Krieger 1959), Troitsky (Troitsky 1987), and Ugural (Ugural 1998). Some commonly encountered cases are given in this section.

Reinforced Concrete Slabs

In a two-way reinforced concrete slab (Fig. 4-7), the bending stiffness constants are usually taken as

$$\left. \begin{aligned} D_x &= \frac{E_c I_x}{1 - \mu_c^2} & D_y &= \frac{E_c I_y}{1 - \mu_c^2} \\ D_{xy} &= \mu_c \sqrt{D_x D_y} & D_s &= \frac{1 - \mu_c}{2} \sqrt{D_x D_y} \\ H &= \sqrt{D_x D_y} \end{aligned} \right\} \quad (4-28)$$

where

E_c = modulus of elasticity of concrete;

E_s = modulus of elasticity of steel;

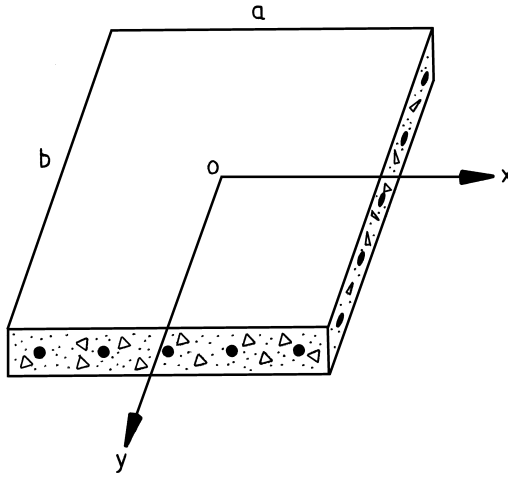


Figure 4-7. Reinforced concrete slab.

I_x = moment of inertia about the neutral axis in an $x = \text{constant}$ direction

$$= (I_{cx} + (n - 1)I_{sx})|_{x = \text{constant}};$$

I_y = moment of inertia about the neutral axis in a $y = \text{constant}$ direction

$$= (I_{cy} + (n - 1)I_{sy})|_{y = \text{constant}};$$

I_{cx}, I_{cy} = moment of inertia of concrete about the neutral axis of composite section in the x - and y -axes, respectively;

I_{sx}, I_{sy} = moment of inertia of steel about the neutral axis of composite section in the x - and y -axes, respectively,

$$n = E_s/E_c;$$

μ_c = Poisson's ratio of concrete.

From Eqs. (4-20) and (4-26) it is seen that $H^2 = D_x D_y$. Hence, Eq. (4-23) reduces to two double roots of magnitude

$$g_1 = g_3 = g$$

$$g_2 = g_4 = -g$$

where

$$g = \frac{m\pi}{a} \sqrt[4]{\frac{D_x}{D_y}}$$

and Eq. (4-26) can be rewritten as

$$w = \sum_{m=1}^{\infty} \left[(C_m + D_m y) e^{-gy} + \frac{P_m}{D_x} \left(\frac{a}{m\pi} \right) \right] \sin \frac{m\pi x}{a}.$$

Corrugated Plate

For this type of construction (Fig. 4-8), the bending stiffnesses are given by

$$\left. \begin{aligned} D_x &= \frac{s}{L} \frac{Et^3}{12(1-\mu^2)} \\ DD_y &= EI \quad D_{xy} = H/2 \\ H &= \frac{L}{s} \frac{Et^3}{12(1+\mu)} \end{aligned} \right\} \quad (4-29)$$

where

$$L = s \left(1 + \frac{\pi^2 f^2}{4s^2} \right)$$

$$I = \frac{f^2 t}{2} \left[1 - \frac{0.81}{1 + 2.5 \left(\frac{f}{2s} \right)^2} \right].$$

Stiffened Plates

For this type of construction (Fig. 4-9), the bending stiffnesses are given by

$$\left. \begin{aligned} D_x &= \frac{EI_x}{h} \quad D_y = \frac{EI_y}{g} \\ D_{xy} &= \frac{G}{3} \left(\frac{d_1 t_1^3}{g} + \frac{d_2 t_2^3}{h} \right) \\ H &= \frac{Et^3}{12(1-\mu^2)} + D_{xy} \end{aligned} \right\} \quad (4-30)$$

where I_x and I_y are the composite moment of inertia of stiffeners and plate in the x - and y -directions, respectively.

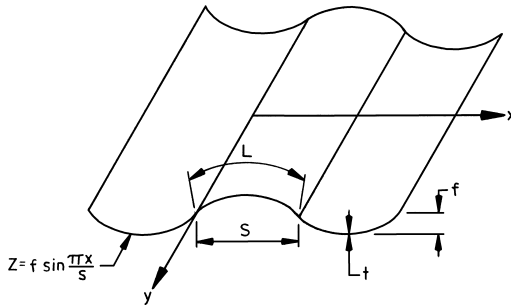


Figure 4-8. Corrugated plate.

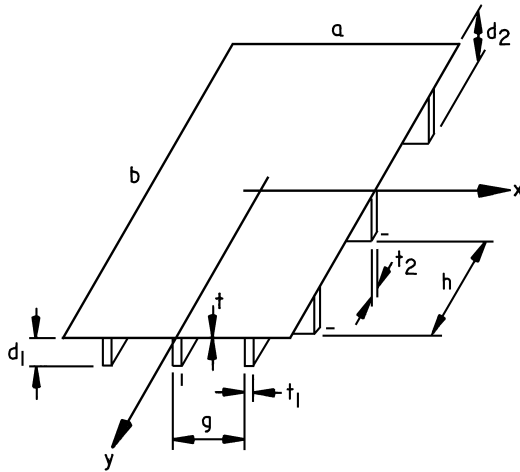


Figure 4-9. Stiffened plate.

Box-Type Bridge Decks

The stiffness values for this type of construction are discussed in detail by Troitsky (Troitsky 1987). The stiffness values of the configuration shown in Fig. 4-10 are given by

$$\left. \begin{aligned} D_x &= \frac{Et^3}{12(1-\mu^2)} \\ D_y &= \frac{EI_B}{g+c} \\ H &= G \frac{2 \left[\frac{1}{2}(g+h)d \right]^2}{\frac{g}{t} + \frac{h+2f}{t_b}} \end{aligned} \right\} \quad (4-31)$$

where

E = modulus of elasticity;

$$G = \frac{E}{2(1+\mu)}$$

I_B = composite stiffness of rib and plate about the neutral axis in the y -direction;

t = thickness of the plate.

The results obtained from Eq. (4-31) tend to be on the unconservative side due to the distortion of the deflected cross section shown in Fig. 4-11. The distortion is due to the flexibility of the span as well as the difference in stiffness between the box plate and deck material. Thus, modifications are needed for the classical solution of Eq. (4-31). This modification, which is beyond the scope of this book, is referred to as the Pelikan-Esslinger method and is detailed by Troitsky.

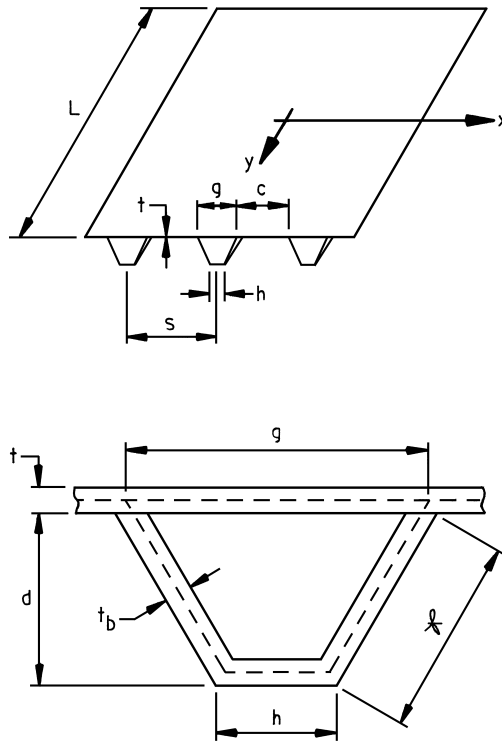


Figure 4-10. Box-type bridge deck.

Problems

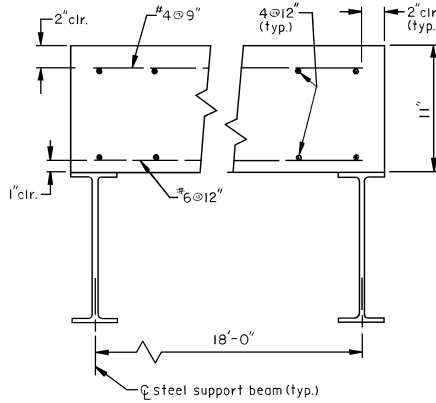
- 4-7 Determine the maximum bending moment in the simply supported concrete slab. The slab is 30 feet long and 18 feet wide. How does this moment compare with ACI's method of calculating the moment as a simply supported unit strip



— — — theoretical deflection line

———— actual deflection line

Figure 4-11. Deflected bridge deck.



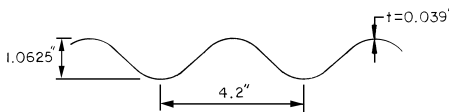
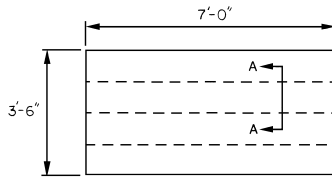
Prob. 4-7. Concrete slab on steel supports.

in the short direction only? Let $p = 200$ psf, $E_s = 30,000$ ksi, $n = 10$, and $\mu_c = 0.15$.

4-8 A corrugated siding for a building is subjected to 40 psf wind load. Determine the maximum bending moment of the panel is assumed simply supported. Let $E = 10,000$ ksi and $\mu = 0.33$.

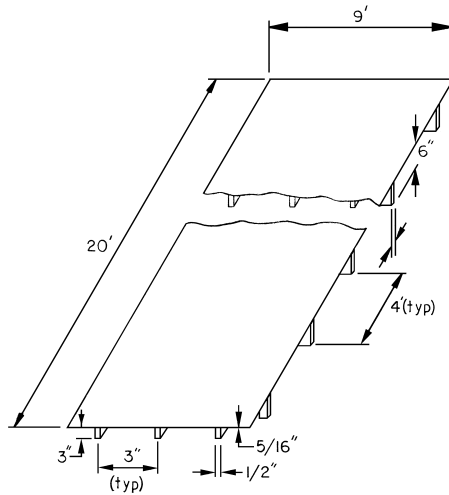
4-9 Part of a ship deck is shown. Determine the maximum bending moment assuming the deck to be simply supported. Let $E = 29,000$ ksi, $p = 75$ psf, and $\mu = 0.30$.

4-10 Refer to Problem 2-8. The side plates have intermediate vertical stiffeners between the bulkheads. The stiffeners are 7C14.75 and spaced on 2 ft centers.



Section A-A

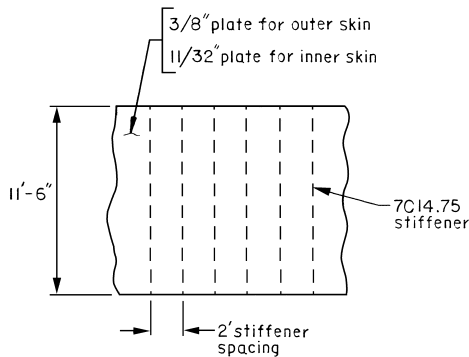
Prob. 4-8. Corrugated siding.



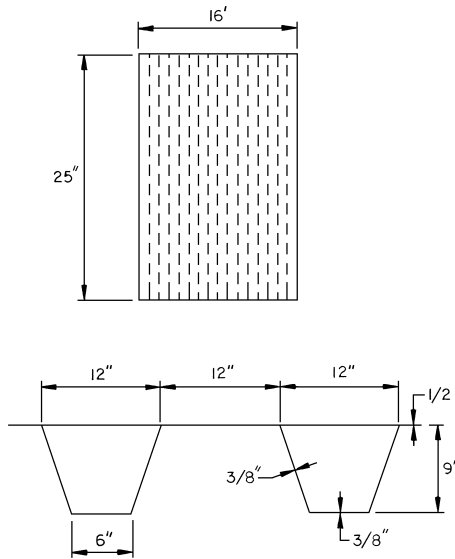
Prob. 4-9. Stiffened ship deck.

The plate thickness is $11/32$ inch. Calculate the maximum stress due to uniform internal pressure of 0.10 psi. Use Eq. (4-26) and let the top and bottom edges be simply supported and the sides fixed.

- 4-11 In Problem 4-10, let the pressure vary as described in Problem 2-8 with a maximum value of 1.25 psi. Calculate the maximum stress.
- 4-12 Oil barges are normally double-hulled due to environmental concerns. Thus, the barge discussed in Problems 2-8, 4-10, and 4-11 consists, in actuality, of an inner and outer skins as shown. How should this sandwich construction be analyzed?



Prob. 4-12. Double hull of oil barge.



Prob. 4-13. Bridge deck.

- 4-13 Find the maximum bending in the box-type bridge deck due to a 200 psf uniform load. Let $E = 30,000$ ksi and $\mu = 0.29$. Assume the deck to be simply supported.

4-6 Design of Plates of Various Shapes and Properties

With the exception of a few cases, the maximum bending of plates with other than rectangular or circular shapes cannot be obtained in a closed form solution based on existing theories. Accordingly, the engineer usually relies on approximate solutions such as those obtained from plastic theory, finite difference, or finite element analysis. Numerous finite element programs are available for personal computer applications. Such programs are used to solve simple plate problems with complicated geometries and boundary conditions.

Factors needed in Eq. (4-21) for solving orthotropic plate theory are presented in this chapter for commonly encountered types and configurations. Factors for plates with configurations and properties other than those discussed in Sections 4-4 and 4-5 or in other references must be obtained experimentally. Calculating these factors as well as the solution of the pertinent differential equation can be made expeditiously by writing a small computer program.

Detailed evaluation of Multilayered and sandwich plates is beyond the scope of this book. Many references are available in the literature on this topic. Szilard

(Szilard 1974), and Ventsel (Ventsel and Krauthammer 2001) give brief description of the behavior of multilayered and sandwich plates.

The methods given in this chapter for determining the maximum bending moment in reinforced concrete slabs have been simplified by ACI for design purposes. ACI-318 uses various factors to approximate the load distribution in the slabs. It then calculates the maximum moment in the slabs by treating them as unit strips in bending and then modify the moments by various factors to approximate orthotropic plate solution.

Approximate Analysis of Plates

5-1 The Strain Energy (Ritz) Method

The classical solutions discussed in the previous chapters are cumbersome, if not impractical, to obtain when the geometry, boundary conditions, and load distribution become more complicated. Other approximate methods are more suitable to solve such problems such as the strain energy (Ritz), yield line theory, finite difference, finite element, and finite strip methods. In this chapter, the Ritz, yield line, and finite difference methods are discussed. The finite element method, which is used for solving complicated geometries, boundary conditions, loading, and physical properties is briefly discussed in Chapter 16. The finite strip method (Cheung 1976) consists of dividing the plate into long strips of narrow width. It was developed, as an alternative method to the finite element method for the purpose of reducing solution time in the 1970's when processing time of computers was relatively slow. Its coverage, however, is beyond the scope of this book.

The Ritz method is well suited for solving plates with complicated boundary conditions, and load distributions that are too difficult to solve by classical methods. It is also a useful tool in parametric studies of such plates where other methods, such as the finite element method, are too cumbersome to utilize. We begin the derivation by stating that the strain energy, U , of the infinitesimal element shown in Fig. 5-1a due to applied stress is obtained from the expression

$$\text{strain energy} = \text{force} \times \text{deflection}.$$

The strain energy, U , due to stress σ_x acting on surfaces $ABCD$ and $A'B'C'D'$ in Fig. 5-1a is given by

$$dU = \int_V \sigma_x d\left(u + \frac{du}{dx}dx\right) dy dz - \int_V \sigma_x du dy dz$$

$$dU = \int_V \sigma_x d\left(\frac{du}{dx}\right) dx dy dz$$

where

u = deflection in the x -axis as shown in Fig. 5-1b;

U = strain energy of a solid element.

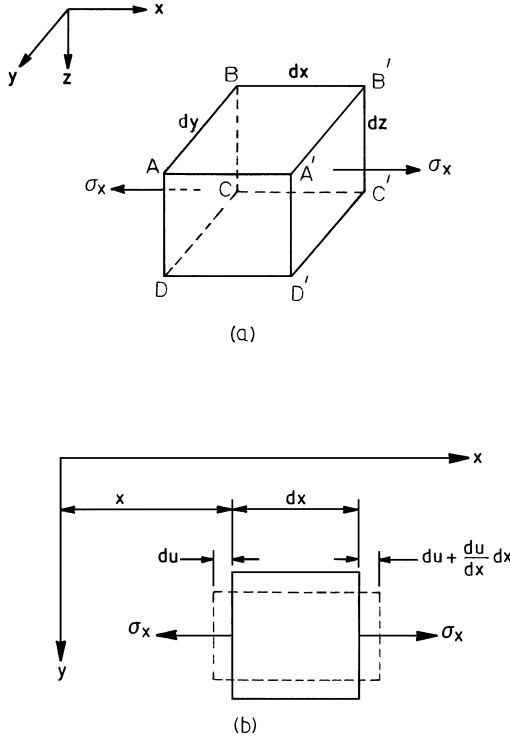


Figure 5-1. Infinitesimal element deformation.

By defining

$$\epsilon_x = \frac{du}{dx}$$

and

$$\epsilon_x = \frac{\sigma_x}{E}$$

the expression for strain energy becomes

$$dU = \int_V \sigma_x d\left(\frac{\sigma_x}{E}\right) dx dy dz$$

$$dU = \frac{\sigma_x^2}{2E} dx dy dz$$

$$dU = \frac{1}{2} \sigma_x \epsilon_x dx dy dz.$$

Summation of the strain energy due to σ_x , σ_y , and σ_z results in

$$U = \int_V \frac{1}{2} (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z) dx dy dz.$$

The strain energy due to shearing stress τ_{xy} is obtained from Fig. 5-2 as

$$dU = \frac{1}{2}(\tau_{xy} dx dz)(\gamma_{xy} dy)$$

$$U = \int_V \frac{1}{2} \tau_{xy} \gamma_{xy} dx dy dz.$$

Hence, the total strain energy due to σ_x , σ_y , σ_z , τ_{xy} , τ_{yz} , and τ_{xz} is given by

$$U = \frac{1}{2} \int_V (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{xz} \gamma_{xz}) dx dy dz.$$

For thin plates,

$$\sigma_z = \tau_{yz} = \tau_{xz} = 0$$

and the strain energy expression reduces to

$$U = \frac{1}{2} \int_V (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy} \gamma_{xy}) dx dy dz.$$

Substituting Eqs. (1-12) and (1-15) into this expression results in

$$U = \frac{D}{2} \int \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \mu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy. \quad (5-1)$$

The external work, W , due to applied loads is given by

$$W = \int p w dx dy. \quad (5-2)$$

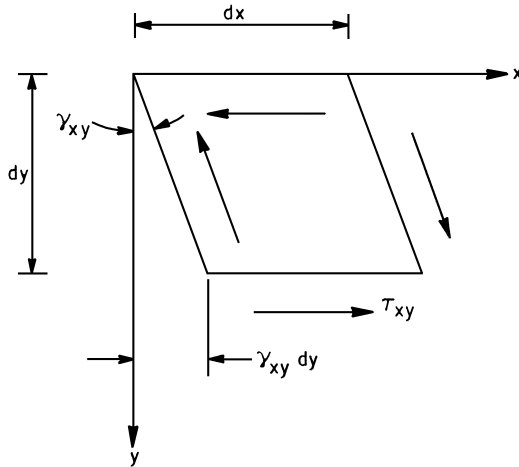


Figure 5-2. Shear deformation.

The total potential energy of the system is defined as

$$\Pi = U - W \quad (5-3)$$

where Π is the total potential energy that must be minimized in order for the plate to be in stable equilibrium.

Equation (5-3) can be solved by expressing the deflection, w , in a geometric series. Ritz (McFarland et al. 1972) suggested a series of the form

$$w = C_1 f_1(x, y) + C_2 f_2(x, y) + \dots \quad (5-4)$$

where the $f(x, y)$ functions represent the deflection of the plate and satisfy the boundary conditions. The constants C are chosen so as to make Eq (5-3) a minimum. Thus

$$\frac{\partial \Pi}{\partial C_1} = 0, \quad \frac{\partial \Pi}{\partial C_2} = 0, \quad \text{etc.} \quad (5-5)$$

Example 5-1

Find the deflection of the simply supported plate shown in Fig. 5-3 due to a uniform pressure p .

Solution

Let the deflection given by Eq. (5-4) be represented by an equation of the form

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (1)$$

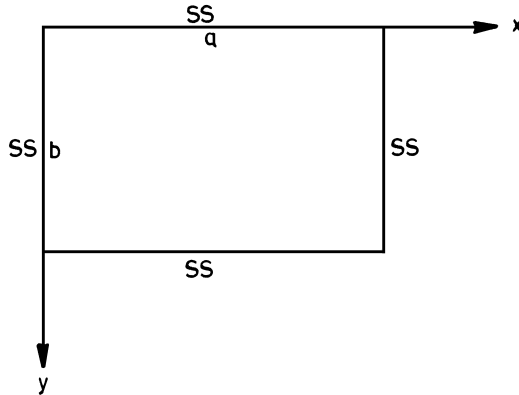


Figure 5-3. Simply supported plate.

which satisfies the boundary conditions. Substituting this expression into Eq. (5-1) and noting that

$$2(1 - \mu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] = 0$$

gives

$$\begin{aligned} U &= \frac{D}{2} \int_0^b \int_0^a \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \left(\frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} \right) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right]^2 dx dy \\ U &= \frac{ab}{8} D \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^2 \left(\frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} \right)^2. \end{aligned} \quad (2)$$

Similarly Eq. (5-2) becomes

$$\begin{aligned} W &= \int_0^b \int_0^a \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p A_{mn} \frac{ab}{\pi^2 mn} (\cos m\pi - 1)(\cos n\pi - 1). \end{aligned} \quad (3)$$

Substituting Eqs. (2) and (3) into Eq. (5-3) gives

$$\Pi = ab \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{D}{8} A_{mn}^2 \left(\frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} \right)^2 - \frac{p}{\pi^2} \frac{A_{mn}}{mn} (\cos m\pi - 1)(\cos n\pi - 1) \right] \quad (4)$$

From Eq. (5-5),

$$\frac{\partial \Pi}{\partial A_{mn}} = 0$$

and

$$A_{mn} = \frac{4p(\cos m\pi - 1)(\cos n\pi - 1)}{D\pi^6 mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2}.$$

Equation (1) for the deflection becomes

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{4p(\cos m\pi - 1)(\cos n\pi - 1)}{D\pi^6 mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \right) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

or

$$w = \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} \frac{16p}{D\pi^6 mn \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}.$$

The solution of rectangular plates by the Ritz method consists of defining an expression in the form of Eq. (5-4) that satisfies all of the boundary conditions. A modification of the Ritz method with Lagrange multipliers, is used to solve plates

with various boundary conditions such as intermediate supports where the expression in Eq. (5-4) cannot satisfy all of the boundary conditions. To begin the derivation let us assume that Eq. (5-4) satisfies all of the boundary conditions except two. Let these two boundary conditions be defined as

$$B_1(C_1, C_2, \dots) = 0$$

and

$$B_2(C_1, C_2, \dots) = 0.$$

Because B_1 and B_2 are the reaction constraints, they must be added to the total work and Eq. (5-3) becomes

$$\Pi = U - W + K_1 B_1 + K_2 B_2 \quad (5-6)$$

where K_1 and K_2 are constants. Minimizing Eq. (5-6) results in the following simultaneous equations

$$\begin{aligned} \frac{\partial \Pi}{\partial C_1} = 0 &= \frac{\partial U}{\partial C_1} - \frac{\partial W}{\partial C_1} + K_1 \frac{\partial B_1}{\partial C_1} + K_2 \frac{\partial B_2}{\partial C_1} \\ \frac{\partial \Pi}{\partial C_2} = 0 &= \frac{\partial U}{\partial C_2} - \frac{\partial W}{\partial C_2} + K_1 \frac{\partial B_1}{\partial C_2} + K_2 \frac{\partial B_2}{\partial C_2} \\ &\vdots \\ \frac{\partial \Pi}{\partial K_1} = 0 &= B_1 \\ \frac{\partial \Pi}{\partial K_2} = 0 &= B_2. \end{aligned}$$

Solution of these simultaneous equations yields the expression for the deflection.

Example 5-2

The plate shown in Fig. 5-4 is simply supported at the edges and is also supported by a column at $x = a/2$ and $y = b/2$. Find the deflection due to a uniform pressure p .

Solution

Let the deflection be given by

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \quad (1)$$

Then from Eq. (4) of Example 5-1

$$\begin{aligned} U - W &= ab \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{D}{8} A_{mn}^2 \left(\frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} \right)^2 \right. \\ &\quad \left. - \frac{p}{\pi^2} \frac{A_{mn}}{mn} (\cos m\pi - 1)(\cos n\pi - 1) \right]. \end{aligned} \quad (2)$$

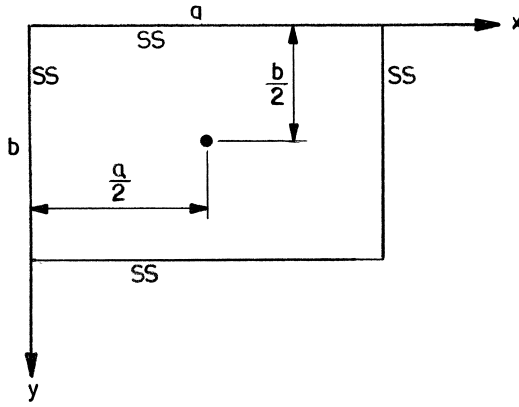


Figure 5-4. Rectangular plate with an internal support.

As Eq. (1) for the deflection is not satisfied at $x = a/2$ and $y = b/2$, it follows that a constraint equation must be expressed as

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi}{2} \sin \frac{n\pi}{2} = 0$$

and the total energy of the system is

$$\Pi = U - W + K_1 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi}{2} \sin \frac{n\pi}{2}$$

where $U - W$ is given by Eq. (2). Minimizing Π with respect to A_{mn} gives

$$\frac{\partial \Pi}{\partial A_{mn}} = 0 = + \frac{Dab\pi^4}{4} [(m/a)^2 + (n/b)^2]^2 A_{mn} - \frac{pab(\cos m\pi - 1)(\cos n\pi - 1)}{\pi^2 mn} + K_1 \sin \frac{m\pi}{2} \sin \frac{n\pi}{2}$$

$$A_{mn} = \frac{4pa^4(\cos m\pi - 1)(\cos n\pi - 1)}{D\pi^6 mn [m^2 + n^2(a/b)^2]^2} + \frac{4a^3 K_1}{D\pi^4 b [m^2 + n^2(a/b)^2]^2} \sin \frac{m\pi}{2} \sin \frac{n\pi}{2}$$

$$\frac{\partial \Pi}{\partial K_1} = 0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi}{2} \sin \frac{n\pi}{2}.$$

Substituting A_{mn} into this equation and solving for K_1 gives

$$K_1 = \frac{pab}{\pi^2} \frac{A_1}{A_2}$$

where

$$A_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\cos m\pi - 1)(\cos n\pi - 1)}{mn [m^2 + n^2(a/b)^2]^2} \sin \frac{m\pi}{2} \sin \frac{n\pi}{2}$$

$$A_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi}{2} \sin \frac{n\pi}{2}}{[m^2 + n^2(a/b)^2]^2}$$

and

$$w = \frac{4pa^4}{D\pi^6} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\cos m\pi - 1)(\cos n\pi - 1)}{mn[m^2 + n^2(a/b)^2]^2} - \frac{A_1}{A_2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \frac{m\pi}{2} \sin \frac{n\pi}{2}}{[m^2 + n^2(a/b)^2]^2} \right\} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}.$$

Problems

5-1 Derive Eq. (5-1).

5-2 Find the deflection of the concrete balcony with an interior support. The uniform load is taken as p . Let $a = 72$ inches, $b = 30$ inches, $p = 100$ psf and

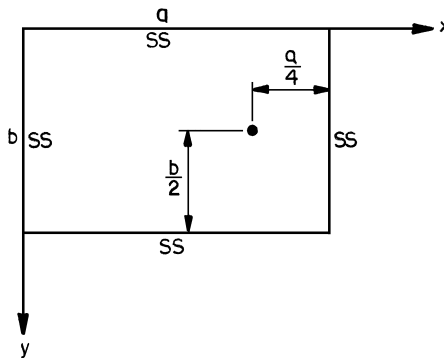
$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}.$$

Let $m, n = 1, 2, 3, 4$.

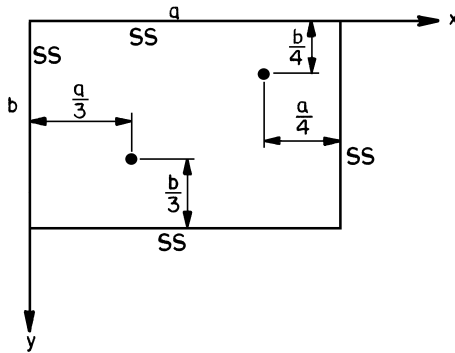
5-3 The steel plate is part of a shoring box for an earth embankment. Find the deflection due to a uniform earth pressure of 120 psf. Let $a = 60$ inches, $b = 75$ inches, and

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}.$$

Let $m, n = 1, 2, 3$.



Prob. 5-2. Simply supported plate with an internal support.



Prob. 5-3. Simply supported plate with two internal supports.

5-2 Yield Line Theory

The yield line theory is a powerful tool for solving many complicated plate problems where an exact elastic solution is impractical to obtain and an approximate solution is acceptable. It is best suited for plates with free boundary conditions and concentrated loads. The theory is based on the assumption that the stress-strain diagram of the material can be idealized as shown in Fig. 5-5a. At point *A* on the elastic stress-strain diagram, the stress distribution across a plate of thickness *t* is as shown in Fig. 5-5b. For a plate under external moment *M*, with no in-plane forces, the equilibrium equation for external and internal moments is

$$M = (\sigma_y)(t/2)(1/2)(2t/3)$$

or

$$\sigma_y = \frac{6M}{t^2}, \quad (5-7)$$

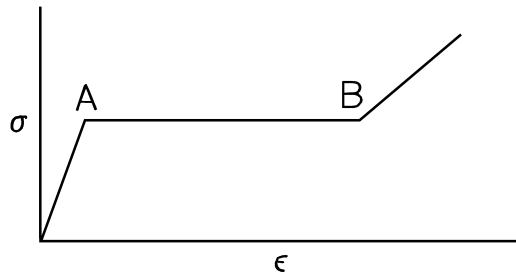
which is the basic relationship for bending stress of an elastic plate. For design purposes the yield stress is divided by a factor of safety to obtain an allowable stress.

As the load increases, the outer fibers of the plate are strained past the yield point *A* in Fig. 5-5a. As the strain approaches point *B*, the stress distribution becomes, for all practical purposes, as shown in Fig. 5-5c. Summation of internal and external moments gives

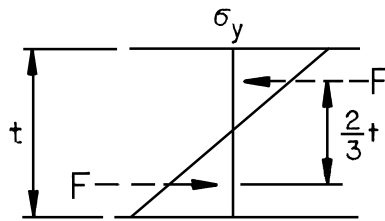
$$M_p = (\sigma_y)(t/2)(t/2)$$

or

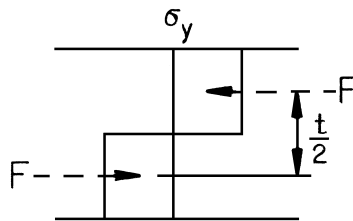
$$\sigma_y = \frac{4M_p}{t^2}. \quad (5-8)$$



(a)



(b)



(c)

Figure 5-5. Elastic and plastic stress distribution.

Equation (5-8) is the basic equation for the bending of a plate in accordance with the plastic theory. For design purposes, the applied loads are multiplied by a load factor to obtain the design loads.

The plastic theory is applied in industry to both metallic plates as well as concrete slabs. Tensile stress in concrete slabs due to plastic moments is resisted by reinforcing bars, and tests conducted on such slabs (Wood 1961) verify the applicability of the plastic theory.

A comparison of Eqs. (5-7) and (5-8) indicates that there is a reduction factor of 1.5 on stress level for a given moment and thickness. The comparison also

shows there is a reduction in thickness by the amount of $(1.5)^{1/2}$ for a given moment and stress level.

Example 5-3

The maximum bending moment in a circular plate subjected to applied loads is 3000 inch-lbs/inch. Calculate the required thickness using elastic and plastic methods. Use a safety factor of 2.0, a load factor of 2.0, and a yield stress of 36,000 psi.

Solution

(a) Elastic Analysis

$$\sigma_a = \text{allowable stress} = 36,000/2.0 = 18,000 \text{ psi}$$

Hence,

$$t = \sqrt{\frac{6M}{\sigma_a}} = \sqrt{\frac{6 \times 3000}{18,000}} = 1.0 \text{ inch.}$$

(b) Plastic Analysis

$$M_p = 3000 \times 2.0 = 6000 \text{ inch-lbs/inch}$$

and

$$t = \sqrt{\frac{4M_p}{\sigma_y}} = \sqrt{\frac{4 \times 6000}{36,000}} = 0.82 \text{ inch.}$$

Hence, a savings of 22% in thickness is obtained by using plastic versus elastic analysis for the same factor of safety.

The yield line theory in plate analysis is very similar to the plastic hinge theory in beam analysis. Application of the plastic hinge theory in beams is illustrated by referring to the fixed beam shown in Fig. 5-6a. The elastic moment diagram due to the concentrated load is shown in Fig. 5-6b. The maximum elastic moment is $4FL/27$ and occurs at the right end support. As the load increases, the moment at the right end reaches M_p and a plastic hinge is developed at that location. However, because the moments at the left end and under the load are less than M_p , the beam can carry more load because it is still statically determinate. Eventually the moment under the load reaches M_p and a plastic hinge is developed there also. However, the beam is still stable and more load can be applied to the beam until the moment at the left side reaches M_p and the beam becomes unstable as shown in Fig. 5-6c. At this instance the moment under the load, M_p , is equal to the moment, M_p , at the ends of the span. The magnitude of the moments can be determined by equating the external work to the internal work. The amount of external work is

$$E.W. = (F)(w)$$

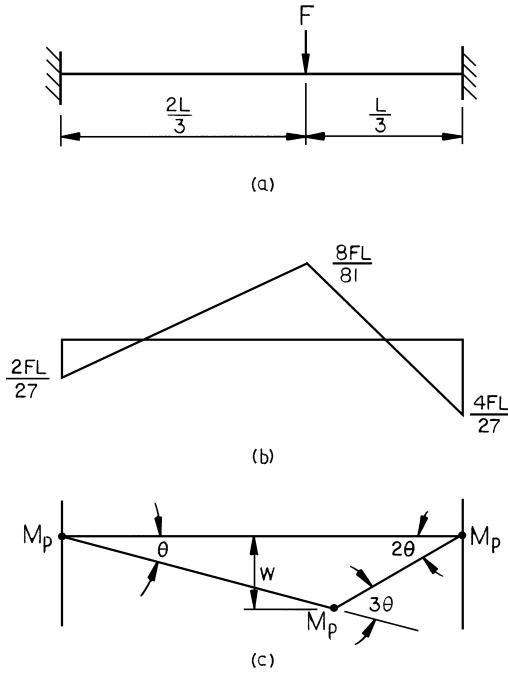


Figure 5-6. Beam with fixed ends.

while the internal work is given by

$$I.W. = M_p(\theta) + M_p(3\theta) + M_p(2\theta).$$

θ can be expressed as

$$\tan\theta \approx \theta = \frac{w}{(2/3)L}$$

or

$$\theta = \frac{3w}{2L}.$$

Hence, the expression for internal work becomes

$$I.W. = 6M_p \frac{3w}{2L}.$$

Equating external and internal work gives

$$M_p = \frac{FL}{9}.$$

The ratio of M_p to M_e is 0.75. This coupled with the fact that $t_e = \sqrt{6M_e/S}$ while $t_p = \sqrt{4M_p/S}$ results in a net ratio of t_p to t_e of 0.7. Thus, a 30% savings in thickness is achieved by plastic analysis of this beam.

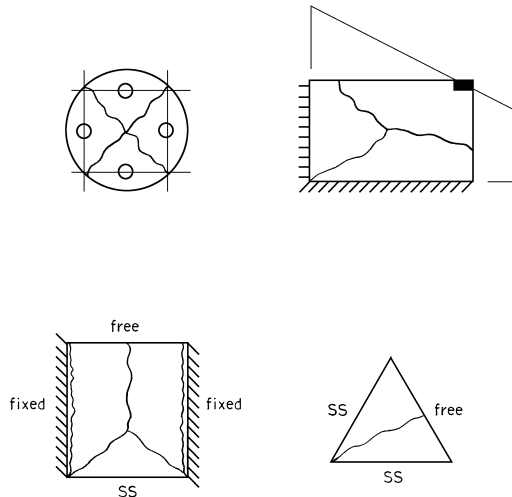


Figure 5-7. Yield lines for various plates.

For plate analysis, the plastic hinges become yield lines. Also, axes of rotation develop in plates rather than points of rotation. Some of the properties of yield lines and axes of rotation are

1. In general, yield lines are straight.
2. Axes of rotation of a plate lie along lines of support.
3. Axes of rotation pass over columns.
4. A yield line passes through the intersection of axes of rotation of adjacent plate segments.

Some illustrations of plates with various geometries, supports, and yield lines are shown in Fig. 5-7. It must be noted that the failure mechanism method described here is an upper bound solution and all failure mechanism patterns must be investigated in order to obtain a safe solution. However, the failure mechanisms for the class of problems discussed here have been verified experimentally and can thus be used for design purposes.

Example 5-4

Find the maximum plastic moment in a simply supported square plate subjected to a uniform load of intensity p .

Solution

The collapse mechanism of the square plate in Fig. 5-8a consists of four yield lines. Section AA through the diagonal of the plate, Fig. 5-8b, details the rotation

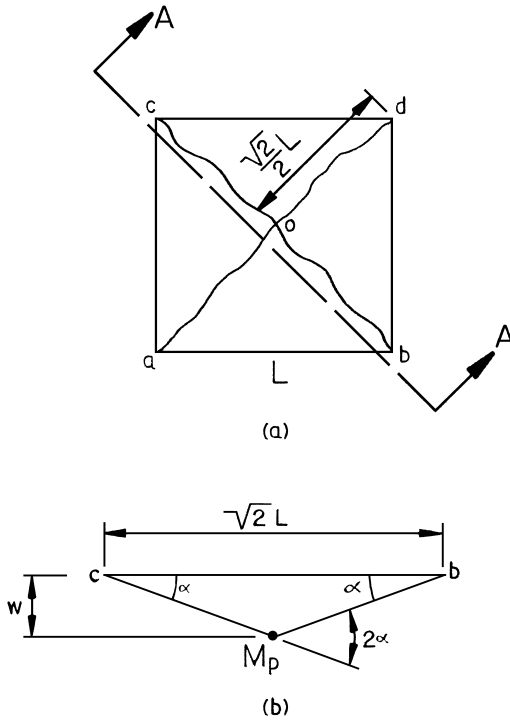


Figure 5-8. Yield lines in a simply supported plate.

of one of the yield lines. The plastic moment, M_p , at this yield line undergoes a rotation of 2α over the length od .

external work = internal work

(p) (volume of pyramid) = $(M_b)(2\alpha)(\text{length})(4 \text{ yield lines})$

$$(p)(L^2 \times w/3) = (M_p)(2\alpha)\left(\frac{\sqrt{2}L}{2}\right)(4).$$

For small deflection w , we let

$$\tan \alpha \approx \alpha = \frac{w}{\sqrt{2}L/2}.$$

We then get

$$M_p = \frac{pL^2}{24}.$$

The analysis method discussed in Example 5-4 becomes cumbersome for complicated geometries. A more efficient method of formulating the expression for internal work is to use vector designations. We illustrate the method by

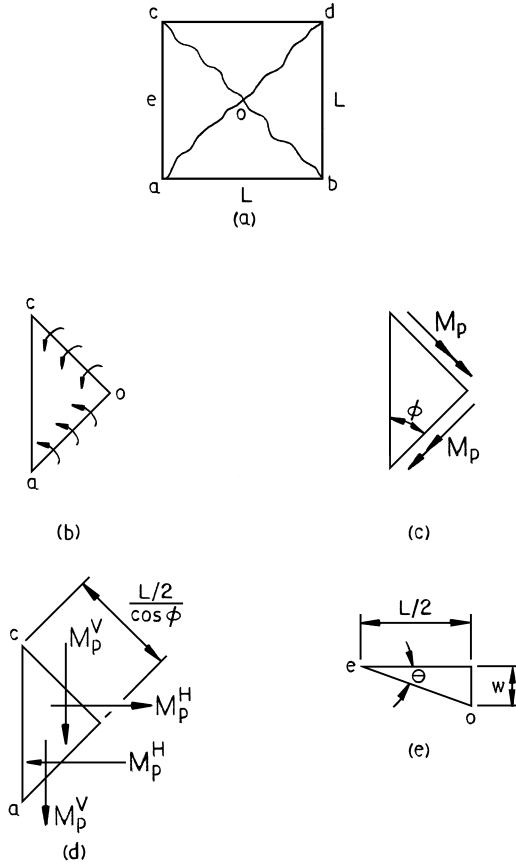


Figure 5-9. Vector representation of plastic moments.

referring to the square plate with the failure mechanism shown in Fig. 5-9a. The applied moments in one panel (Fig. 5-9b) can be designated by the vectors given in Fig. 5-9c. The horizontal components of the vector moment (Fig. 5-9d) cancel each other while the vertical vectors are additive.

If we use the approach discussed in Example 5-4, then from sketches (d) and (e) of Fig. 5-9 we get

$$\begin{aligned}
 I.W. \text{ for one panel} &= 2(M_p^V) \left(\frac{L/2}{\cos \phi} \right) (\theta) \\
 &= 2(M_p \cos \phi) \left(\frac{L/2}{\cos \phi} \right) \left(\frac{w}{L/2} \right) \\
 &= 2(M_p)w.
 \end{aligned}$$

This same result can be obtained more efficiently by observing in sketches (d) and (e) of Fig. 5-9 that the product of the quantity M_p times the slanted length is of the same magnitude as the product of the moment M_p applied along the edge L . Hence,

$$\begin{aligned} I.W. \text{ for one panel} &= (M_p)(L)(\theta) \\ &= (M_p)(L)\left(\frac{w}{L/2}\right) \\ &= 2(M_p)w. \end{aligned}$$

As the rotation of the outside edge is obtained more easily than the rotation of the inner yield lines, the vector approach will be used in all subsequent discussions.

Example 5-5

Find the required thickness of a square plate fixed at the edges and subjected to a uniform load of 12 psi. Use a load factor of 2.0, and yield stress of 36 ksi.

Solution

From Fig. 5-10a,

$$E.W. = I.W.$$

$$\begin{aligned} \left(\frac{1}{3}\right)(20^2)(2.0 \times 12)w &= \text{work due to } M_p \text{ at internal yield lines} \\ &+ \text{work due to } M_p \text{ at edges} \end{aligned}$$

and from Fig. 5-10b,

$$3200 w = 4(M_p)(L)\left(\frac{w}{L/2}\right) + 4(M_p)(L)\left(\frac{w}{L/2}\right)$$

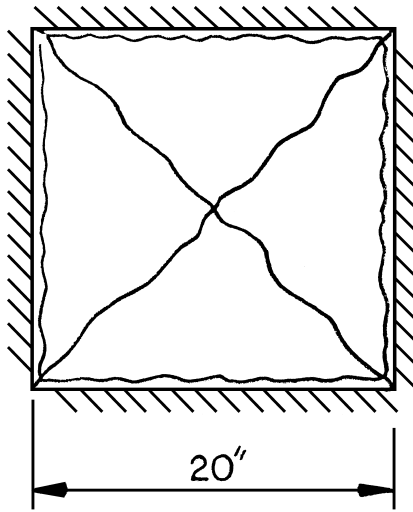
or,

$$M_p = 200 \text{ inch-lbs/inch.}$$

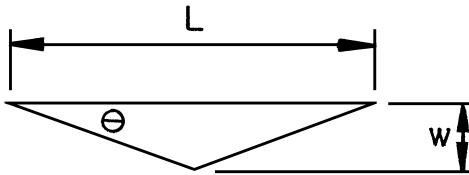
$$t = \sqrt{\frac{4M_p}{\sigma_y}} = \sqrt{\frac{4 \times 200}{36,000}} = 0.15 \text{ inch.}$$

Example 5-6

Find M_p of the rectangular plate in Fig. 5-11a due to uniform load p . The plate is simply supported.



(a)



(b)

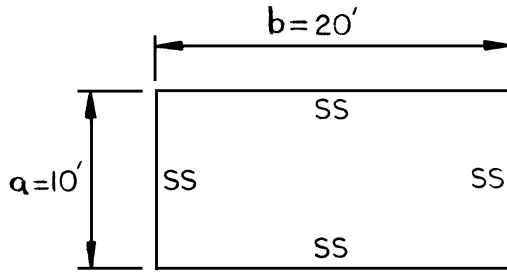
Figure 5-10. Square plates fixed along the edges.*Solution*

The yield lines take the shape shown in Fig. 5-11b. Distance x is unknown and must be determined.

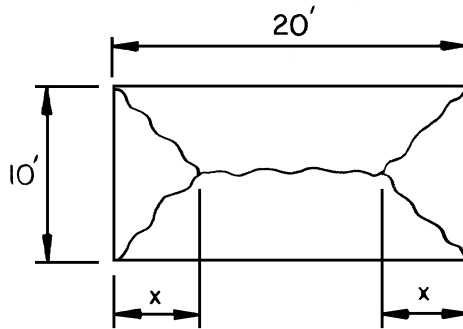
$$I.W. = 2(M_p)(a)\left(\frac{w}{x}\right) + 2(M_p)(b)\left(\frac{w}{a/2}\right)$$

From Fig. 5-11c,

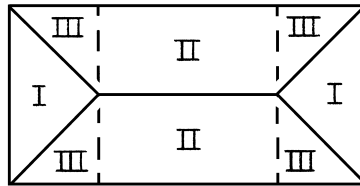
$$\begin{aligned} E.W. &= p(E.W. \text{ I} + E.W. \text{ II} + E.W. \text{ III}) \\ &= p\left[2\left(\frac{ax}{2}\right)(1/3) + 2(b-2x)(a/2)(1/2) + 4(x)(a/2)(1/2)(1/3)\right]w. \end{aligned}$$



(a)



(b)



(c)

Figure 5-11. Simply supported rectangular plate.

Equating internal and external work and solving for M_p gives

$$M_p = p \left[\frac{\frac{ab}{2} - \frac{ax}{3}}{2 \left(\frac{2b}{a} + \frac{a}{x} \right)} \right]. \quad (1)$$

Minimizing M_p by taking its derivative with respect to x and equating it to zero gives

$$x^2 + \frac{a^2}{b}x - \frac{3a^2}{4} = 0. \quad (2)$$

For $a = 10$ ft, and $b = 20$ ft, Eq. (2) gives $x = 6.51$ ft. From Eq. (1),

$$\begin{aligned} M_p &= p \left(\frac{14,400 - 3124.8}{2(4.00 + 1.54)} \right) \\ &= 1018 \, p \text{ inch-lbs/inch.} \end{aligned}$$

Problems

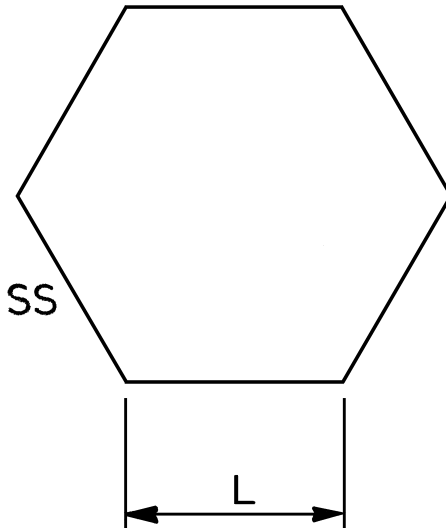
5-4 Find M_p due to a uniform load on the simply supported hexagon plate.

5-5 Find M_p in the skewed bridge slab due to uniform load p .

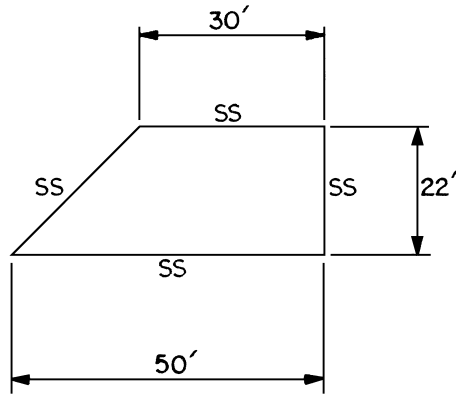
5-6 Determine M_p in the triangular weir plate due to uniform load p . The plate is fixed at all edges.

5-3 Further Application of the Yield Line Theory

The deflections obtained from the yield line theory are larger than those obtained from the elastic theory due to reduced thicknesses. This should be considered in applications where small deflections are critical to the performance of equipment



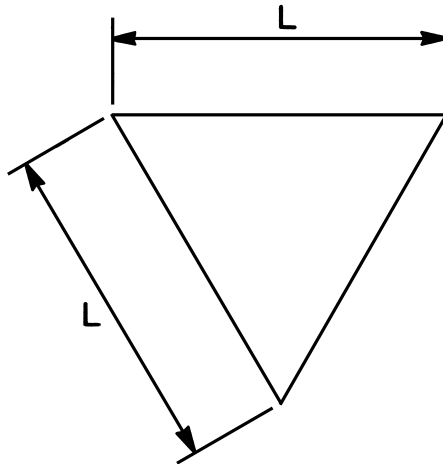
Prob. 5-4. Hexagonal plate.

**Prob. 5-5.** Trapezoidal plate.

such as flanges and other sealing components. Also, the yield line theory tends to give an upper bound solution. Accordingly, all possible yield line paths must be investigated in order to obtain a true solution. This is especially important in plates with free edges as illustrated in Example 5-7.

Example 5-7

Find the maximum moment in a triangular section, Fig. 5-12a, subjected to a uniform load p .

**Prob. 5-6.** Triangular plate.

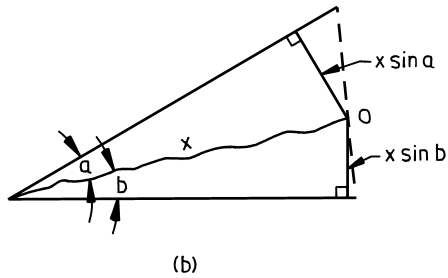
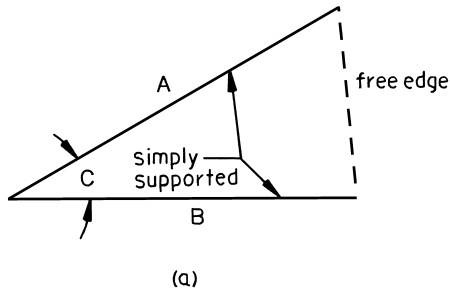


Figure 5-12. Plate with a free edge.

Solution

Assume the yield line to be as shown in Fig. 5-12b at an angle a from side A . Also, assume w to be the deflection at point O . Then

$$C = a + b$$

$$\begin{aligned} I.W. &= M_p(x \cos a) \frac{w}{x \sin a} + M_p(x \cos b) \frac{w}{x \sin b} \\ &= M_p(\cot a + \cot b) w \end{aligned}$$

$$\begin{aligned} E.W. &= p \left[\frac{(A)(x \sin a)}{2} \frac{1}{3} + \frac{(B)(x \sin b)}{2} \frac{1}{3} \right] w \\ &= \frac{px}{6} (A \sin a + B \sin b) w. \end{aligned}$$

Equating internal and external work results in

$$M_p = \frac{px}{a} \frac{A \sin a + B \sin b}{\cot a + \cot b}.$$

Minimizing M_p with respect to a results in

$$a = C/2$$

which indicates that in any triangular plate with one edge free, the yield line always bisects the angle between the two simply supported edges. The plastic moment is given by

$$M_p = \frac{px}{6} \frac{A \sin (C/2) + B \sin (C/2)}{\cot (C/2) + \cot (C/2)}.$$

Where,

$$x = \frac{K_1}{\sin(C/2)} \sin K_2$$

$$K_1 = 0.5(A^2 + B^2 - 2AB \cos C)^{1/2}$$

$$K_2 = 180 - (C/2 + K_3)$$

$$K_3 = \sin^{-1} \left(\frac{2B}{2K_1} \sin C/2 \right)$$

Example 5-8

Find the maximum moment in a uniformly loaded square plate, Fig. 5-13a, with three sides simply supported and one side free.

Solution

(a) Let the failure pattern be as shown in Fig. 5-13b. Let maximum deflection w be at point 0. Then,

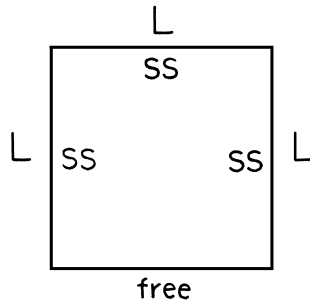
$$\begin{aligned} I.W. &= 2M_p(L) \frac{w}{L/2} + M_p(L) \frac{w}{L-y} \\ &= \left(4M_p + \frac{M_p L}{L-y} \right) w \\ E.W. &= p \left[\frac{L(L-y)}{2} \frac{1}{3} + \frac{Ly}{2} + 2(L-y) \frac{L}{2} \frac{1}{2} \frac{1}{3} \right] w \\ &= p \left[\frac{L(L-y)}{3} + \frac{Ly}{2} \right] w. \end{aligned}$$

Equating internal and external work gives

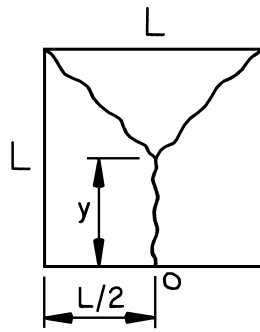
$$M_p = \frac{pL}{6} \frac{2L^2 - Ly - y^2}{5L - 4y}.$$

Minimizing M_p with respect to y gives

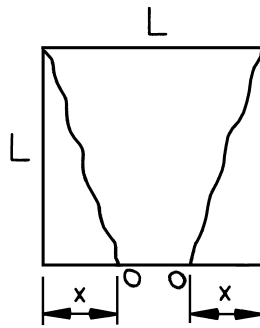
$$y = 2.15L \text{ which is discarded}$$



(a)



(b)



(c)

Figure 5-13. Square plate with one side free.

or

$$y = 0.35L$$

and

$$M_p = \frac{pL^2}{14.14}.$$

(b) Let the failure pattern be as shown in Fig. 5-13c. Let maximum deflection w be along points 0-0. Then,

$$\begin{aligned} I.W. &= 2M_p L \frac{w}{x} + 2M_p x \frac{w}{x} \\ &= \left(\frac{2M_p L}{x} + \frac{2M_p x}{L} \right) w \\ E.W. &= p \left[\frac{2Lx}{2} \frac{1}{3} + \frac{2Lx}{2} \frac{1}{3} + (L - 2x)L \frac{1}{2} \right] w \\ &= p \left[\frac{2Lx}{3} + \frac{(L - 2x)L}{2} \right] w. \end{aligned}$$

Equating internal and external work results in

$$M_p = \frac{pL}{12} \frac{3L^2x - 2x^2L}{L + x}.$$

Minimizing M_p with respect to x gives

$$x = -1.87L \text{ or } x = 0.54L \text{ which are impossible.}$$

Thus, the first alternate (a) controls.

The yield line mechanism in circular plates subjected to uniform or concentrated loads have a radial pattern. The failure mode is conical in shape as illustrated in Example 5-9. This circular pattern must also be investigated for rectangular plates under concentrated loads.

Example 5-9

Find the moment in a simply supported circular plate due to (a) uniform load p and (b) concentrated load F in the middle.

Solution

(a) From Fig. 5-14 using w as the deflection in the middle,

$$\begin{aligned} I.W. &= \int_0^{2\pi} M_p a \, d\phi \frac{w}{a} \\ &= 2\pi M_p w \\ E.W. &= (\pi a^2 p / 3) w \end{aligned}$$

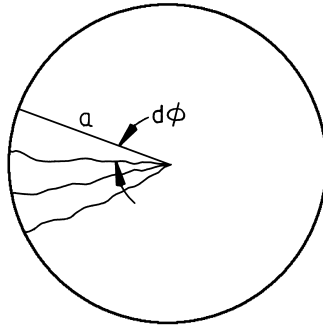


Figure 5-14. Yield lines in a circular plate.

and

$$M_p = pa^2/6.$$

(b) Again from Fig. 5-14 using w as the deflection in the middle,

$$E.W. = F(w)$$

and

$$M_p = \frac{F}{2\pi}.$$

It is of interest to note that M_p in this case is independent of the radius.

Example 5-10

Find the moment in a square plate subjected to a concentrated load F in the middle if the plate is (a) simply supported on all sides or (b) fixed on all sides.

Solution

(a) From Figure 5-15a, using a straight line mechanism with a deflection, w , in the middle

$$4(M_p)(a) \frac{w}{a/2} = F(w)$$

$$M_p = 0.125F.$$

From Figure 5-15b, using a circular mechanism with a deflection, w , in the middle

$$2(2\pi M_p)w = F(w)$$

$$M_p = 0.08F.$$

Hence, the straight line mechanism controls.

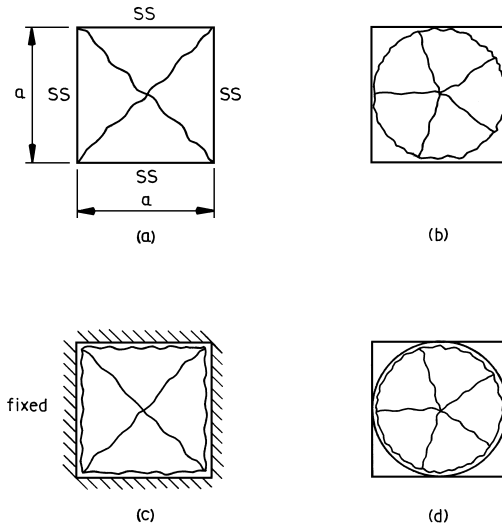


Figure 5-15. Yield lines in a square plate.

(b) From Figure 5-15c, using a straight line mechanism with a deflection, w , in the middle

$$2 \left[4(M_p)(a) \frac{1}{a/2} \right] w = F(w)$$

$$M_p = 0.063F.$$

From Figure 5-15d, using a circular mechanism with a deflection, w , in the middle

$$2(2\pi M_p) w = F(w)$$

$$M_p = 0.08F$$

and the circular mechanism controls.

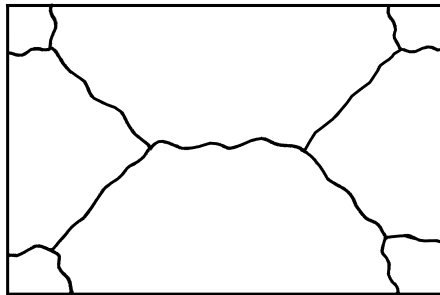


Figure 5-16. Yield lines in a reinforced concrete slab.

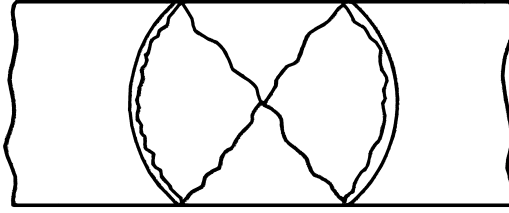


Figure 5-17. Fan-like yield lines in long narrow plates.

A possible failure mechanism that occurs near corners of reinforced concrete slabs is shown in Fig. 5-16. Experimental (Moy 1981) and theoretical evaluation (Jones 1966) showed that for square slabs without corner reinforcement, this mechanism results in a maximum moment that is about 10% higher than that obtained from a straight line mechanism. For slabs with small acute corner angles and without adequate corner reinforcement, the maximum moment is about 25% higher than that obtained from straight line mechanisms. Thus for reinforced concrete slabs with sharp corners, the plastic moments developed previously must be increased accordingly.

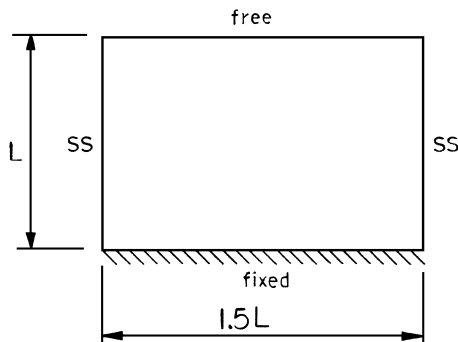
Another failure mechanism that occurs in long narrow plates is shown in Fig. 5-17 and should be investigated together with other failure modes.

Problems

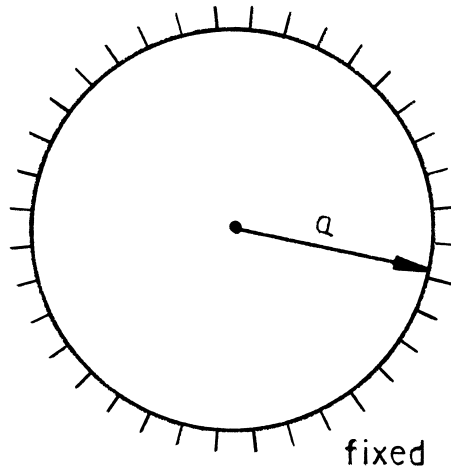
5-7 Find M_p in the rectangular base plate due to uniform load p .

5-8 Find M_p in Problem 5-7 if a concentrated load F is applied at the middle of the free edge.

5-9 The internal vessel tray is assumed fixed at the outer edge. Determine M_p due to a concentrated load in the middle.



Prob. 5-7. Rectangular plate with various boundary conditions.



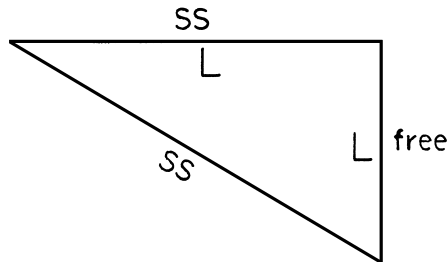
Prob. 5-9. Circular plate fixed at edge.

5-10 Determine M_p in the triangular plate shown due to uniform load p .

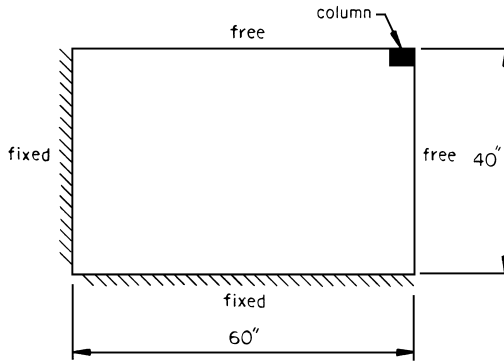
5-11 Find M_p in the rectangular balcony with two adjacent sides fixed and the opposite corner supported by a column. The balcony is uniformly loaded with 75 psf. How will reinforcing bars be arranged.

5-4 Finite Difference Method

The finite difference method is well suited for solving plate problems with irregular boundary conditions and non-uniform load distribution where a classical or other approximate solution is too complicated to obtain. In the finite difference method, the differential equations are replaced by a number of approximate equations that are solved simultaneously. Figure 5-18a represents a portion of a plate subdivided into a finite grid system. Points surrounding point o are



Prob. 5-10. Triangular plate.



Prob. 5-11. Balcony supported by a column at one corner.

designated as east, west, north, or south as illustrated in the Figure. The slope of any point o in the x -direction is shown in Fig. 5-18b. This slope can be approximated by variety of equations. One such equation is called the first forward differential equation and is expressed by

$$\frac{\partial w}{\partial x} = \frac{1}{\lambda}(w_E - w_o) \quad (5-9)$$

where w is the deflection at a grid point and λ is the grid spacing. The slope can also be expressed by a first backward differential equation as

$$\frac{\partial w}{\partial x} = \frac{1}{\lambda}(w_o - w_W) \quad (5-10)$$

or by the central differential equation as

$$\frac{\partial w}{\partial x} = \frac{1}{2\lambda}(w_E - w_W) \quad (5-11)$$

The second derivative of the deflection in the x -direction is obtained from any of the above equations. However, for best accuracy it is advantageous to obtain an expression where the resultant terms are as close to the point under consideration as possible. Accordingly, taking the first backward derivative of the first forward differential Eq. (5-9) best approximates the second derivative

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= \frac{1}{\lambda} \left[\frac{\partial w_E}{\partial x} - \frac{\partial w_o}{\partial x} \right] \\ &= \frac{1}{\lambda} \left[\frac{w_E - w_o}{\lambda} - \frac{w_o - w_W}{\lambda} \right] \end{aligned}$$

or,

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{\lambda^2}(w_E - 2w_o + w_W) \quad (5-12)$$

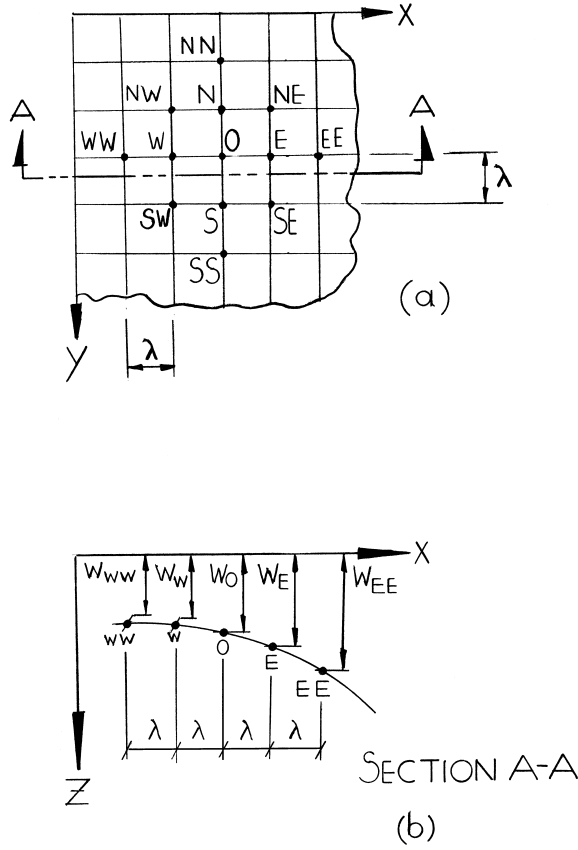


Figure 5-18. Finite difference grid system.

The third derivative in the x-direction is obtained by taking the central derivative of the above equation. This yields

$$\begin{aligned}
 \frac{\partial^3 w}{\partial x^3} &= \frac{1}{\lambda^2} \left[\frac{\partial w_E}{\partial x} - 2 \frac{\partial w_O}{\partial x} + \frac{\partial w_W}{\partial x} \right] \\
 &= \frac{1}{\lambda^2} \left[\frac{w_{EE} - w_O}{2\lambda} - 2 \left(\frac{w_E - w_W}{2\lambda} \right) + \frac{w_O - w_{WW}}{2\lambda} \right] \\
 \frac{\partial^3 w}{\partial x^3} &= \frac{1}{2\lambda^3} \left[w_{EE} - 2w_E + 2w_W - w_{WW} \right] \quad (5-13)
 \end{aligned}$$

The fourth derivative in the x-direction is obtained from

$$\frac{\partial^4 w}{\partial x^4} = \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 w}{\partial x^2} \right)$$

$$\begin{aligned}
&= \frac{\partial^2}{\partial x^2} \left[\frac{1}{\lambda^2} (w_E - 2w_o + w_W) \right] \\
\frac{\partial^4 w}{\partial x^4} &= \frac{1}{\lambda^4} [w_{EE} - 4w_E + 6w_o - 4w_W + w_{WW}] \quad (5-14)
\end{aligned}$$

Similarly, in the y-direction,

$$\frac{\partial w}{\partial y} = \frac{1}{2\lambda} (w_S - w_N) \quad (5-15)$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{\lambda^2} (w_S - 2w_o + w_N) \quad (5-16)$$

$$\frac{\partial^3 w}{\partial y^3} = \frac{1}{2\lambda^3} [w_{SS} - 2w_S + 2w_N - w_{NN}] \quad (5-17)$$

$$\frac{\partial^4 w}{\partial y^4} = \frac{1}{\lambda^4} [w_{SS} - 4w_S + 6w_o - 4w_N + w_{NN}] \quad (5-18)$$

Mixed partials are derived as follows

$$\begin{aligned}
\frac{\partial^2 w}{\partial x \partial y} &= \frac{\partial}{\partial y} \left(\frac{\partial w}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{w_E - w_W}{2\lambda} \right) \\
\frac{\partial^2 w}{\partial x \partial y} &= \frac{1}{4\lambda^2} (w_{NE} - w_{SE} - w_{NW} + w_{SW}) \quad (5-19)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial^4 w}{\partial x^2 \partial y^2} &= \frac{\partial^2}{\partial y^2} \left(\frac{\partial^2 w}{\partial x^2} \right) = \frac{\partial^2}{\partial y^2} \left(\frac{w_E - 2w_o + w_W}{\lambda^2} \right) \\
\frac{\partial^4 w}{\partial x^2 \partial y^2} &= \frac{1}{\lambda^4} (w_{NE} - 2w_E + w_{SE} - 2w_N + 4w_o - 2w_S + w_{NW} + 2w_W + w_{SW}) \quad (5-20)
\end{aligned}$$

Substituting the above finite difference equations into the differential equation for a plate

$$\nabla^4 w = p(x, y)/D \quad (5-21)$$

results in

$$\begin{aligned}
&20w_o - 8(w_N + w_S + w_E + w_W) + 2(w_{NE} + w_{SE} + w_{NW} + w_{SW}) \\
&+ (w_{WW} + w_{NN} + w_{SS} + w_{EE}) = p\lambda^4/D \quad (5-22)
\end{aligned}$$

This equation is represented in Fig. 5-19. Similarly, the expressions for the moments are obtained from

$$M_x = -(D/\lambda^2) [-(2 + 2\mu)w_o + w_W + w_E + \mu(w_N + w_S)] \quad (5-23)$$

$$M_y = -(D/\lambda^2) [-(2 + 2\mu)w_o + w_N + w_S + \mu(w_E + w_W)] \quad (5-24)$$

$$M_{xy} = [D(1 - \mu)/4\lambda^2] [w_{SE} - w_{NE} - w_{SW} + w_{NW}] \quad (5-25)$$

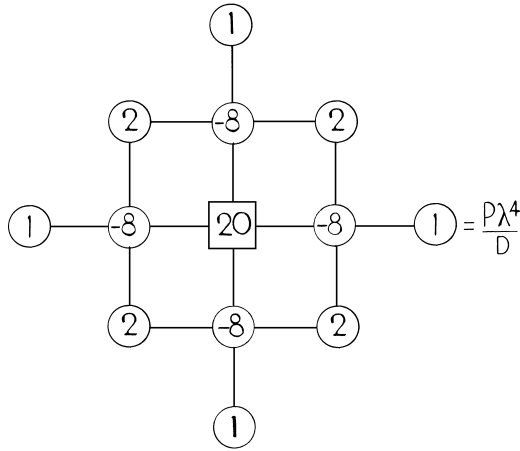


Figure 5-19. Graphical representation of Eq. (5-22).

Example 5-11

A square plate with fixed edges, Fig. 5-20a, is subjected to a uniform load p . Find the maximum deformation and moment by the finite difference method if a) $\lambda = 10$ inches and b) $\lambda = 5$ inches. Let $\mu = 0.3$.

Solution

a) The grid layout, $\lambda = a/2$, and deformation shape in the x -direction is shown in Fig. 5-20b. The deformation shape must be extended past the boundary of the plate in order to solve the equations for nodal points that are close to the boundary. The deformation shape is assumed symmetric about the boundary for the specified fixed boundary conditions. The deformation shape would have been non-symmetric for simply supported boundary conditions.

From Eq. (5-22)

$$20w_o - 8(0 + 0 + 0 + 0) + 2(0 + 0 + 0 + 0) + (w_o + w_o + w_o + w_o) = p\lambda^4/D$$

$$24w_o = p\lambda^4/D$$

or,

$$\begin{aligned} w_o &= 0.042 p\lambda^4/D \\ &= 0.0026 pa^4/D \end{aligned}$$

This result compares with a theoretical deflection value of $w_o = 0.00126 pa^4/D$. Thus, the finite difference method is off by about 100% based on taking only one

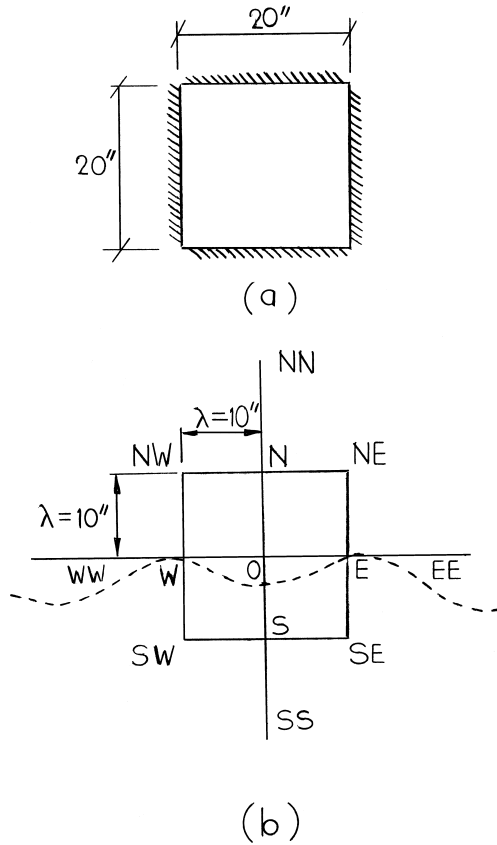


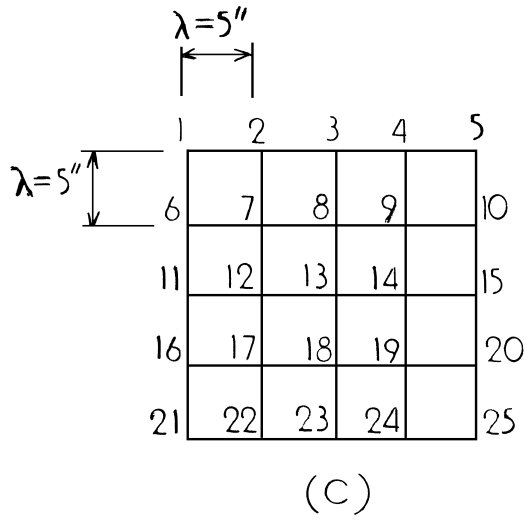
Figure 5-20. Finite difference layout of a square plate.

node at the center of the plate. The accuracy improves significantly as more nodal points are selected.

b) The grid layout, $\lambda = a/4$, is shown in Fig. 5-20c. The nodal points are designated by numbers rather than letters in order to simplify the terminology. Due to symmetry, and knowing that the deformations at the boundary lines are zero, the nodal points can be reduced from 25 to 3 as shown in Fig. 5-20d. Accordingly, only three equations are needed; one for each of the designated nodal points. From Eq. (5-22),
at node 3

$$20w_3 - 8(w_2 + w_2 + w_2 + w_2) + 2(w_1 + w_1 + w_1 + w_1) + (0 + 0 + 0 + 0) = p\lambda^4/D$$

$$20w_3 - 32w_2 + 8w_1 = p\lambda^4/D \quad (1)$$



1	2		1
2	3		2
1	2		1

(d)

Figure 5-20. (continued)

at node 2

$$\begin{aligned}
 20w_2 - 8(0 + w_1 + w_1 + w_3) + 2(0 + 0 + w_2 + w_2) + (w_2 + 0 + w_2 + 0) \\
 = p\lambda^4/D \\
 -8w_3 + 26w_2 - 16w_1 = p\lambda^4/D
 \end{aligned} \tag{2}$$

at node 1

$$\begin{aligned}
 20w_1 - 8(w_2 + 0 + 0 + w_2) + 2(0 + 0 + 0 + w_3) + (w_1 + w_1 + w_1 + w_1) \\
 = p\lambda^4/D \\
 2w_3 - 16w_2 + 24w_1 = p\lambda^4/D
 \end{aligned} \tag{3}$$

The three equations can be arranged in matrix form as

$$\begin{bmatrix} 20 & -32 & 8 \\ -8 & 26 & -16 \\ 2 & -16 & 24 \end{bmatrix} \begin{bmatrix} w_3 \\ w_2 \\ w_1 \end{bmatrix} = (p\lambda^4/D) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

from which we obtain

$$w_3 = 0.4607 p\lambda^4/D = 0.0018 pa^4/D$$

$$w_2 = 0.3090 p\lambda^4/D = 0.0012 pa^4/D$$

$$w_3 = 0.2093 p\lambda^4/D = 0.00082 pa^4/D$$

The maximum deformation calculated in this case is much closer to the theoretical value than that calculated in part a). The maximum moment in the middle of the plate is obtained from Eq. (5-23) as

$$M_x = -(D/\lambda^2)[-(2 + 2\mu)w_3 + w_2 + w_2 + \mu(w_2 + w_2)]$$

Substituting the values of w_3 and w_2 into this equation gives

$$M_x = 0.3944 p\lambda^2 = 0.0247 pa^2$$

The theoretical moment is $M_x = 0.0264 pa^2$.

The maximum moment at the edge is obtained from Eq. (5-23) as

$$M_x = -(D/\lambda^2)[-(2 + 2\mu)0 + w_2 + w_2 + \mu(0 + 0)]$$

Substituting the value of w_2 into this equation gives

$$M_x = 0.6180 p\lambda^2 = 0.0386 pa^2$$

This moment compares with a theoretical moment of $M_x = 0.0513 pa^2$. Again, the accuracy can be improved by increasing the number of nodal points.

In the above example the assumed deformation shape, Fig. 5-20b, was extended beyond the boundary of the plate since the deformation expression for nodal points close to the boundary includes terms that fall outside the boundary. Another approach is to modify the deformation equation of such

nodal points so that only nodal points within the boundaries are referenced. This is illustrated in Fig. 5-21. In Fig. 5-21a, nodal point *o* falls near a corner of a fixed-fixed boundary. Nodal points *W*, *NW*, *SW*, *N*, and *NE* must be set to zero since they fall along the boundary. The slope in the *x*-direction of point *W*, which falls on the boundary is zero and from Eq. (5-11)

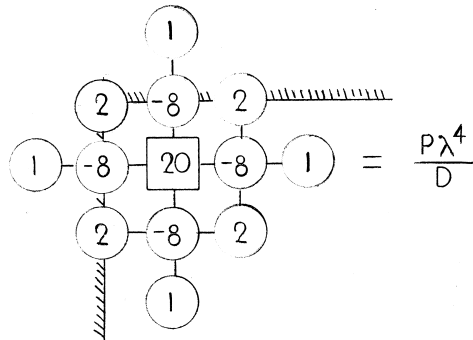
$$\frac{\partial w}{\partial x} = \frac{1}{2\lambda}(w_{WW} - w_o) = 0$$

or,

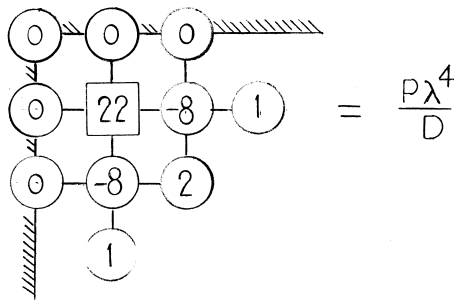
$$w_{WW} = w_o$$

Similarly for point *N* in the *y*-direction,

$$\frac{\partial w}{\partial y} = \frac{1}{2\lambda}(w_{NN} - w_o) = 0$$



(a)



(b)

Figure 5-21. Point near a corner of a plate.

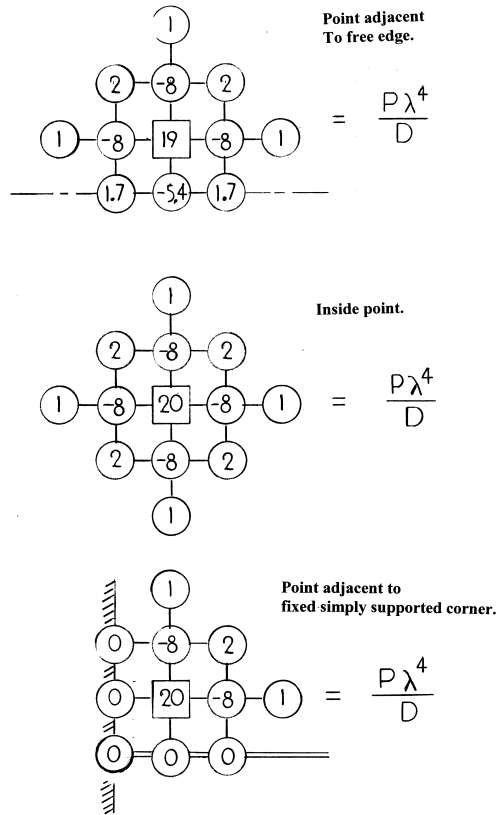


Figure 5-22. (continued)

The accuracy of the solution in the above example was shown to improve with a mesh system requiring smaller values of λ . However, the convergence of Eq. (5-22) and the moment equations, which are based on linear approximations, is normally slow. An improved solution with faster convergence is attained by using Taylor expansions (Szilard, 1974). The deflection Eq. (5-22) takes on the form shown in Fig. 5-23. The moment expressions given by Eqs. (5-23) through (5-25) are then modified accordingly. The solution, however, requires handling a large number of simultaneous equations.

Problems

- 5-12 Solve Example 5-11 using simply supported edges.
 5-13 Solve Example 5-11 using Fig. 5-22 for points near the edges.

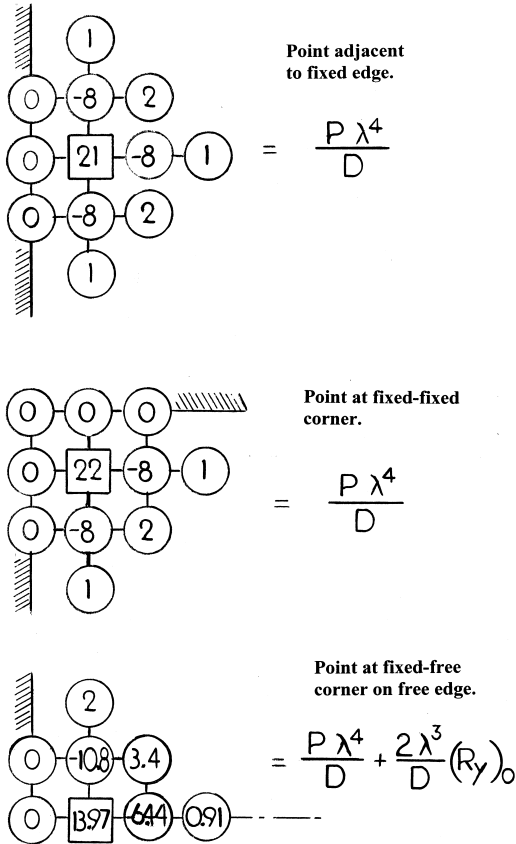


Figure 5-22. (continued)

5-5 Design Concepts

The Ritz method is an excellent tool for designing plates that cannot easily be solved by the classical theory. Plates with internal or partial edge supports are ideally suited for solving by the Ritz method. The approximations needed to obtain an answer are usually adequate for most design applications.

The plastic theory is used as an approximation for determining maximum moments in plates that cannot be solved easily by the classical plate theory due to complex geometry, boundary conditions, or applied loading. In many structures this approximation is adequate for design purposes. When a more accurate analysis is needed, the plastic theory is used first as an approximation followed by a more rigorous analysis such as the finite element method. Table 5-1 lists the

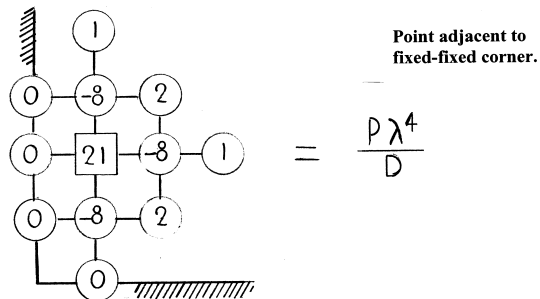
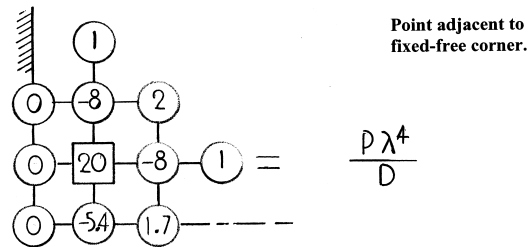
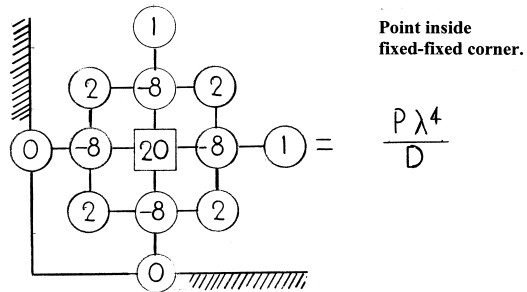


Figure 5-22. (continued)

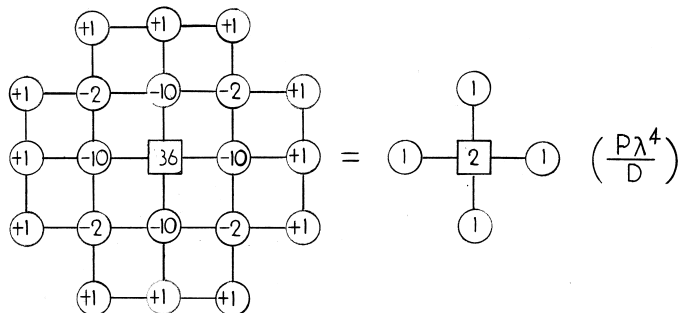


Figure 5-23. Taylor series expansion of plate differential equation.

Table 5-1. Plastic bending moments in various plates

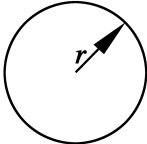
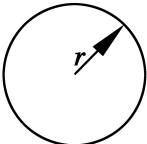
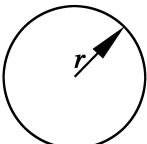
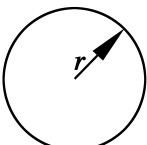
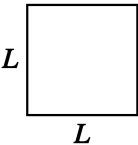
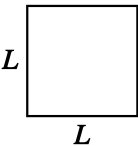
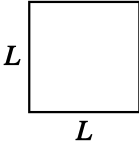
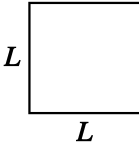
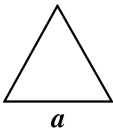
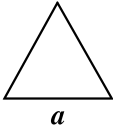
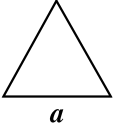
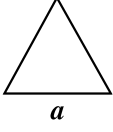
Case	Maximum Moment
1. Uniform load, p , simply supported edge	$M_p = \frac{pr^2}{6}$
	
2. Uniform load, p , fixed edge	$M_p = \frac{pr^2}{12}$
	
3. Concentrated load, F , in the middle simply supported edge	$M_p = \frac{F}{2\pi}$
	
4. Concentrated load, F , in the middle fixed edge	$M_p = \frac{F}{4\pi}$
	
5. Uniform load, p , simply supported edges	$M_p = \frac{pL^2}{24}$
	
6. Uniform load, p , fixed edges	$M_p = \frac{pL^2}{48}$
	

Table 5-1. (continued)

7. Concentrated load, F , in the middle simply supported edges	$M_p = \frac{F}{8}$
	
8. Concentrated load, F , in the middle fixed edges	$M_p = \frac{F}{4\pi}$
	
9. Uniform load, p , simply supported edges	$M_p^a = \frac{pa^2}{72}$
	
10. Uniform load, p , fixed edges	$M_p^a = \frac{pa^2}{144}$
	
11. Concentrated load, F , at the centroid simply supported edges	$M_p^a = \frac{F}{6\sqrt{3}}$
	
12. Concentrated load, F , at the centroid fixed edges	$M_p^a = \frac{F}{12\sqrt{3}}$
	

^aThese bending moments are to be increased by a factor of 1.25 to account for sharp angle effect at the corners of reinforced concrete slabs.

maximum plastic moment for some frequently encountered plates with various loading and boundary conditions. It should be noted that the deflection due to plastic design is larger than that obtained from the elastic theory due to reduced thickness. Thus, extra precaution must be given to the design of components that cannot tolerate large deflections such as cover plates in flanged openings.

Equation (5-8) for plastic bending of a plate is used by numerous international codes to establish an upper limit on the allowable bending stress values. The ratio obtained by dividing Eq. (5-7) by Eq. (5-8) is 1.5 and is referred to as the shape factor. It indicates that for a given bending moment, plastic analysis of plates results in a stress level that is 50% lower than that determined from the elastic theory for the same factor of safety. Accordingly, many standards such as the ASME Boiler and Pressure Vessel Code use an allowable stress for plates in bending that is 50% higher than the tabulated allowable membrane stress value.

The finite difference method is an excellent design tool for solving plates that are irregular in shape, have internal supports, or have complex boundary conditions. The solution is easily obtained by solving a set of simultaneous equations and the accuracy of the answer can readily be verified by decreasing the mesh size and comparing the results.

Buckling of Plates

6-1 Circular Plates

When a thin elastic plate is subjected to compressive in-plane axial loads, in conjunction with small applied lateral loads or imperfections in the plate, the in-plane deflections increase gradually with an increase in the applied loads up to a certain critical point. Beyond this point a slight increase in axial loads causes a large and sudden increase in the deflection. This phenomenon, called buckling, is the subject of this chapter for circular and rectangular plates. A more comprehensive treatment of this subject is given by Timoshenko (Timoshenko and Gere, 1961), Bloom (Bloom and Coffin, 2001), Szilard (Szilard, 1974), and Iyengar (Iyengar, 1988).

The differential equation for the bending of a circular plate subjected to lateral loads, p , is obtained from Eq. (3-11) as

$$r^2 \frac{d^2 \phi}{dr^2} + r \frac{d\phi}{dr} - \phi = -\frac{Qr^2}{D}.$$

Where,

$$D = Et^3/12(1 - \mu^2)$$

E = modulus of elasticity

Q = shear

r = radius of plate

t = thickness

ϕ = angle as shown in Fig. 6-1.

μ = Poisson's ratio

When in-plane forces N_r are applied as shown in Fig. 6-1, and the lateral loads, p , are reduced to zero, then the corresponding value of Q is

$$Q = N_r \phi.$$

Letting

$$A^2 = N_r/D \tag{6-1}$$

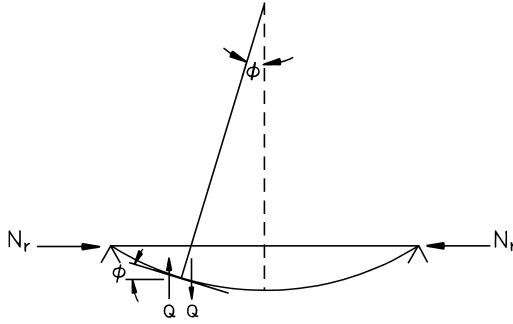


Figure 6-1. In-plane load in a circular plate.

the differential equation becomes

$$r^2 \frac{d^2 \phi}{dr^2} + r \frac{d\phi}{dr} - (r^2 A^2 - 1) \phi = 0.$$

Defining

$$x = Ar \quad \text{and} \quad dx = A dr \quad (6-2)$$

we get

$$x^2 \frac{d^2 \phi}{dx^2} + x \frac{d\phi}{dx} + (x^2 - 1) \phi = 0. \quad (6-3)$$

The solution of this equation is in the form of a Bessel function. From Eq. (B-3) of Appendix B,

$$\phi = C_1 J_1(x) + C_2 Y_1(x) \quad (6-4)$$

at $r = 0$, $Y_1(x)$ approaches infinity. Hence, C_2 must be set to zero and Eq. (6-4) becomes

$$\phi = C_1 J_1(x). \quad (6-5)$$

For a fixed boundary condition, $\phi = 0$ at $r = a$ and a nontrivial solution of Eq. (6-5) is

$$J_1(x) = 0$$

or from Table B-1 of Appendix B, $x = 3.83$, and Eqs. (6-1) and (6-2) give

$$\sqrt{N_r/D}(a) = 3.83$$

or

$$N_{cr} = \frac{14.67D}{a^2}. \quad (6-6)$$

For a simply supported plate, the moment at the boundary $r = a$ is zero and Eqs. (10-5) and (3-4) give

$$N_{cr} = \frac{4.20D}{a^2}. \quad (6-7)$$

Equations (6-6) and (6-7) are for the critical buckling load of circular plates with fixed and simply supported boundary conditions, respectively. It is of interest to note that contrary to column buckling where the column is rendered ineffective in carrying any further loads subsequent to buckling, a simply supported circular plate is capable of carrying additional loads in the postbuckling phased due to in-plane biaxial membrane stress (Bloom and Coffin 2001). Postbuckling analysis of such plates (Sherbourne 1961) results in an equation of the form

$$N_u/N_{cr} = 1 + 0.241(w/t)^2 \quad (6-8)$$

Where,

N_u = ultimate axial load

N_{cr} = critical buckling load

t = plate thickness

w = plate deflection

This equation is independent of the ratio r/t and is limited to $w/t < 3.0$. Values greater than 3.0 result in plastic strains in the plate.

Example 6-1

What is the required thickness of a simply supported circular plate subjected to a lateral pressure of 2 psi and in-plane compressive force of 100 lb/inch if $a = 29$ inches, $\mu = 0.31$, $E = 30,000$ ksi, allowable stress in bending = 10,000 psi, and factor of safety ($F.S.$) for buckling = 3.0.

Solution

From Example 3-1,

$$M = \frac{Pa^2}{16}(3 + \mu) = 348.0 \text{ inch-lb/inch}$$

$$t = \sqrt{6 \times 348/10,000} = 0.46 \text{ inch.}$$

Try $t = 0.50$ inch.

$$\text{actual bending stress} = \frac{6 \times 348}{0.5^2} = 8350 \text{ psi}$$

$$\text{actual compressive stress} = 100/0.5 = 200 \text{ psi}$$

$$D = \frac{30,000 \times 0.5^3}{12(1 - 0.31^2)} = 345,720 \text{ inch-lbs.}$$

From Eq. (6-7),

$$\sigma_{cr} = \frac{4.20 \times 345,720}{29^2 \times 0.5} = 3450 \text{ psi.}$$

allowable compressive stress = $3450/3 = 1150$ psi.

Using the interaction equation

$$\frac{\text{actual bending stress}}{\text{allowable bending stress}} + \frac{\text{actual compressive stress}}{\text{allowable compressive stress}} \leq 1.0$$

gives

$$\frac{8350}{10,000} + \frac{200}{1150} = 0.83 + 0.17 = 1.0 \text{ o.k.}$$

Use $t = 0.50$ inch.

The governing differential equation for the buckling of circular plates with a central hole, i.e. annular plate, is derived much the same way as that for solid circular plates. However, the general differential equation (Bloom and Coffin 2001) for an annular plate is more complicated than a solid plate. The equation is similar to Eq.(3-21) for the general bending of a plate with the exception of having applied axial in-plane loads N_r and tangential in-plane loads N_θ rather than lateral loads p . Shearing forces $N_{\theta r}$ are assumed to be zero. The equation is expressed as

$$D \nabla^4 w = -N_r \frac{\partial^2 w}{\partial r^2} - N_\theta \left(\frac{1}{r} \frac{\partial^2 w}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right). \quad (6-9)$$

where, $\nabla^4 w$ is given by

$$\begin{aligned} \nabla^4 w = & \frac{\partial^4 w}{\partial r^4} + \frac{2}{r} \frac{\partial^3 w}{\partial r^3} - \frac{1}{r^2} \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^3} \frac{\partial w}{\partial r} + \frac{2}{r^2} \frac{\partial^4 w}{\partial r^2 \partial \theta^2} - \frac{2}{r^3} \frac{\partial^3 w}{\partial \theta^2 \partial r} \\ & + \frac{4}{r^4} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r^4} \frac{\partial^4 w}{\partial \theta^4} \end{aligned} \quad (6-10)$$

The solution of Eq. (6-9) for specific plate geometry with the appropriate boundary conditions results in four simultaneous equations involving Bessel functions. Setting the determinate of the coefficients of these four equations to zero and solving the resultant polynomial provides the eigenvalues associated with buckling of the specific plate and boundary conditions. The critical buckling load is obtained from the lowest eigenvalue. Table 6-1 lists the critical buckling load for an annular plate subjected to in-plane compressive loads at the outside edge. The inner edge is free and the outer edge is either simply supported or fixed.

Table 6-1. Critical Buckling Coefficient, k , of an Annular plate with a Free Inside Edge.

b/a	k values	
	Simply supported outer edge	Fixed outer edge
0.0	4.20	14.64
0.2	3.60	13.32
0.4	2.76	17.76
0.6	2.28	Note 1
0.8	2.04	Note 1

$$N_{cr} = kD/a^2$$

Where,

a = outer radius

b = inside radius

$D = Et^3 / 12(1 - \mu^2)$

E = modulus of elasticity

k = constant

N_{cr} = critical load, force/length

t = thickness

μ = Poisson's ratio

Note 1: the critical load increases exponentially (Timoshenko and Gere, 1961)

Problems

6-1 Derive Eq. (6-7).

6-2 What is the required thickness of a fixed circular plate subjected to a lateral pressure of 3 psi and in-plane compressive force of 500 lb/inch if $a = 40$ inches, $\mu = 0.30$, $E = 25,000$ ksi, allowable stress in bending = 25,000 psi, and factor of safety ($F.S.$) for buckling = 4.0?

6-3 What is the effect of N_r on the bending moments if it were in tension rather than compression?

6-2 Rectangular Plates

Buckling of rectangular plates is most commonly caused by in-plane shear or in-plane axial loads in the x and y directions. In this and the following sections classical as well as numerical methods are presented for solving these loading conditions. Other loading conditions due to temperature gradients and support settlements are beyond the scope of this book.

The differential equation for the bending of a rectangular plate with lateral load, p , is given by Eq. (1-26) as

$$\nabla^4 w = p/D. \quad (6-11)$$

If the plate is additionally loaded in its plane (Fig. 6-2a) then summation of forces in the x -direction gives

$$N_x dy + N_{yx} dx - \left(N_x + \frac{\partial N_x}{\partial x} dx \right) dy - \left(N_{yx} + \frac{\partial N_{yx}}{\partial y} dy \right) dy = 0$$

or

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0. \quad (6-12)$$

Similarly, summation of forces in the y -direction gives

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0. \quad (6-13)$$

Summation of forces in the z -direction (Fig. 6-2b) gives the following for N_x :

$$-N_x dy \frac{\partial w}{\partial x} + \left(N_x + \frac{\partial N_x}{\partial x} dx \right) \left(\frac{\partial w}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial x} \right) dx \right) dy$$

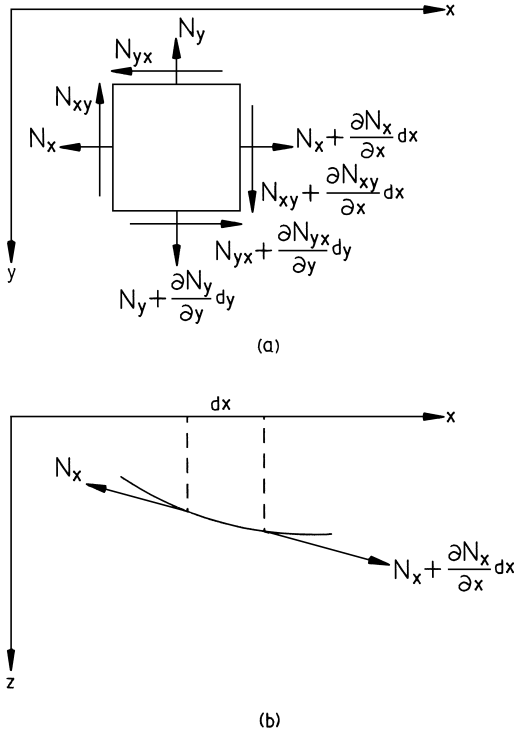


Figure 6-2. In-plane loads in a rectangular plate.

which reduces to

$$N_x \frac{\partial^2 w}{\partial x^2} dx dy + \frac{\partial N_x}{\partial x} \frac{\partial w}{\partial x} dx dy. \quad (6-14)$$

Similarly, for N_y ,

$$N_y \frac{\partial^2 w}{\partial y^2} dx dy + \frac{\partial N_y}{\partial y} \frac{\partial w}{\partial y} dx dy. \quad (6-15)$$

For N_{xy} from Fig. 6-3,

$$-\left(N_{xy} dy \frac{\partial w}{\partial y}\right) + \left(N_{xy} + \frac{\partial N_{xy}}{\partial x} dx\right) \left(\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial x \partial y} dx\right) dy$$

or

$$N_{xy} \frac{\partial^2 w}{\partial x \partial y} dx dy + \frac{\partial N_{xy}}{\partial x} \frac{\partial w}{\partial y} dx dy. \quad (6-16)$$

Similarly for N_{yx}

$$N_{yx} \frac{\partial^2 w}{\partial x \partial y} dx dy + \frac{\partial N_{yx}}{\partial x} \frac{\partial w}{\partial x} dx dy. \quad (6-17)$$

The total sum of Eq. (6-11), which was obtained by summing forces in the z -direction, with Eqs. (6-14), (6-15), (6-16), and (6-17) gives the basic differential equation of a rectangular plate subjected to lateral and in-plane loads.

$$\nabla^4 w = \frac{1}{D} \left(p + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \quad (6-18)$$

It should be noted that Eqs. (6-12) and (6-13), which were obtained by summing forces in the x - and y -directions, were not utilized in Eq. (6-18). They

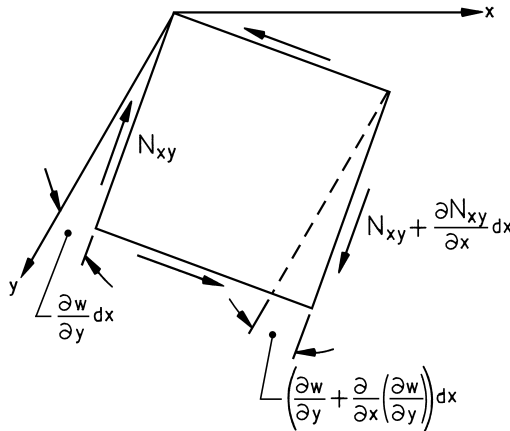


Figure 6-3. In-plane shear loads.

are used to formulate large-deflection theory of plates which is beyond the scope of this book.

Another equation that is frequently utilized in buckling problems is the energy equation. It was shown in Chapter 5 that the strain energy expression due to lateral loads, p , is given by Eq. (5-1) as

$$U = \frac{D}{2} \int_A \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \mu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy. \quad (6-19)$$

The strain energy expression for the in-plane loads is derived from Fig. 6-4 which shows the deflection of a unit segment dx . Hence,

$$dx' = \sqrt{dx^2 - \left(\frac{\partial w}{\partial x} dx \right)^2}$$

$$\epsilon_x = dx - dx' = \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 dx$$

or per unit length,

$$\epsilon_x = \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2.$$

Similarly,

$$\epsilon_y = \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2.$$

It can also be shown that

$$\gamma_{xy} = \frac{\partial w}{\partial x} \frac{\partial w}{\partial y}.$$

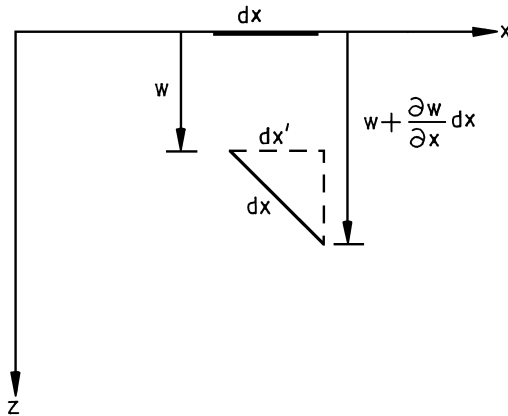


Figure 6-4. Deflection of a unit segment.

Hence, the strain energy expression for the in-plane forces is given by

$$U = \int_A (N_x \epsilon_x + N_y \epsilon_y + N_{xy} \gamma_{xy}) dx dy$$

$$U = \frac{1}{2} \int_A \left[N_x \left(\frac{\partial w}{\partial x} \right)^2 + N_y \left(\frac{\partial w}{\partial y} \right)^2 + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dx dy. \quad (6-20)$$

The total strain energy expression for rectangular plates loaded laterally and in-plane is the summation of expressions (6-19) and (6-20). Thus,

$$U = \frac{D}{2} \int_A \left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2(1 - \mu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] \right\} dx dy$$

$$+ \frac{1}{2} \int_A \left[N_x \left(\frac{\partial w}{\partial x} \right)^2 + N_y \left(\frac{\partial w}{\partial y} \right)^2 + 2N_{xy} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] dx dy. \quad (6-21)$$

The total potential energy of a system is given by Eq. (5-3) as

$$\Pi = U - W \quad (6-22)$$

where W is external work. In order for the system to be in equilibrium, Eq. (6-22) must be minimized.

Example 6-2

Find the buckling stress of a simply supported rectangular plate, Fig. 6-5, subjected to forces N_x .

Solution

Let the deflection be expressed as

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b},$$

which satisfies the boundary conditions.

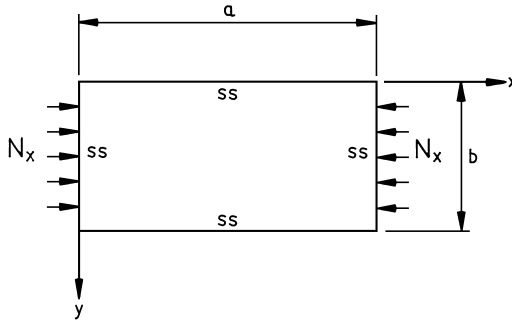


Figure 6-5. Rectangular plate subjected to force N_x .

Substituting this expression into Eq. (6-21), and noting that the expression

$$2(1 - \mu) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] = 0,$$

gives

$$U = \frac{D}{2} \int_0^b \int_0^a \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[A_{mn}^2 \left(\frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} \right)^2 \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} \right] dx dy \\ + \frac{1}{2} \int_0^b \int_0^a \left[(-N_x) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^2 \frac{m^2 \pi}{a^2} \times \sin^2 \frac{m\pi x}{a} \sin^2 \frac{n\pi y}{b} \right] dx dy$$

or,

$$U = \frac{\pi^4 ab}{8} D \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 - \frac{\pi^2 b}{8a} N_x \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 A_{mn}^2. \quad (1)$$

Since there are no lateral loads, we can take the external work in the z -direction as zero and Eq. (6-22) becomes

$$\Pi = U.$$

Solving Eq. (1) for

$$\frac{\partial \Pi}{\partial A_{mn}} = 0$$

we get

$$N_{cr} = \frac{\pi^2 a^2 D}{m^2} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right). \quad (2)$$

The smallest value of Eq. (2) is for $n = 1$. Also, if we substitute

$$\sigma_{cr} = N_{cr}/t$$

and

$$D = \frac{Et^3}{12(1 - \mu^2)}$$

we get

$$\sigma_{cr} = \frac{\pi^2 E}{12(1 - \mu^2)(b/t)^2} K \quad (3)$$

where

$$K = \left(\frac{m}{a/b} + \frac{a/b}{m} \right)^2. \quad (4)$$

A plot of Eq. (4) is shown in Fig. 6-6 and shows that the minimum value of K is 4.0.

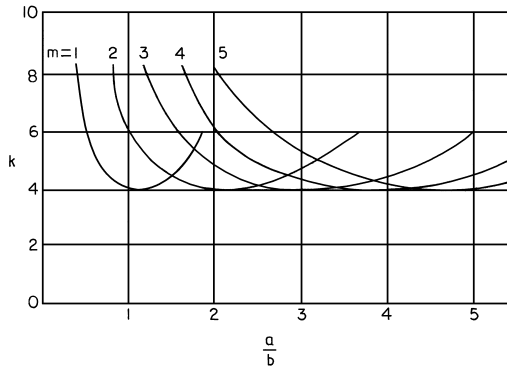


Figure 6-6. Plot of Eq. (4) in Example 6-2.

Equation (3) in Example 6-2 gives the critical elastic buckling stress in a simply supported rectangular plate. The minimum value when $k = 4.0$ is

$$\sigma_{cr} = \frac{\pi^2 E}{3(1 - \mu^2)(b/t)^2} \quad (6-23)$$

This equation can also be written as

$$\sigma_{cr} = K(D/t)(\pi/b)^2 \quad (6-24)$$

where,

b = short dimension of plate

$D = Et^3/12(1 - \mu^2)$

E = modulus of elasticity

$K = 4.0$

t = thickness of plate

σ_{cr} = critical buckling stress of plate

Theoretical postbuckling equations (Bloom and Coffin 2001) as well as experimental work has shown that the critical buckling stress in rectangular plates increases significantly from that of Eq. (6-24) as the buckling deflection increases. This is demonstrated in Fig. 6-7 as a function of the ratio of plate deflection, δ , to thickness t .

Experiments on rectangular plates, Fig. 6-8a, have shown that parts of the plate closer to the edges, Fig. 6-8b, carry significantly more load than at the central portion of the plate. Various investigators developed different expressions for the effective width when ultimate loads are used. Marguerre's (Marguerre, 1937) equation is based on the terminology of Fig. 6-8c and is given by

$$b_e = b(\sigma_{cr}/\sigma_u)^{1/3} \quad (6-25)$$

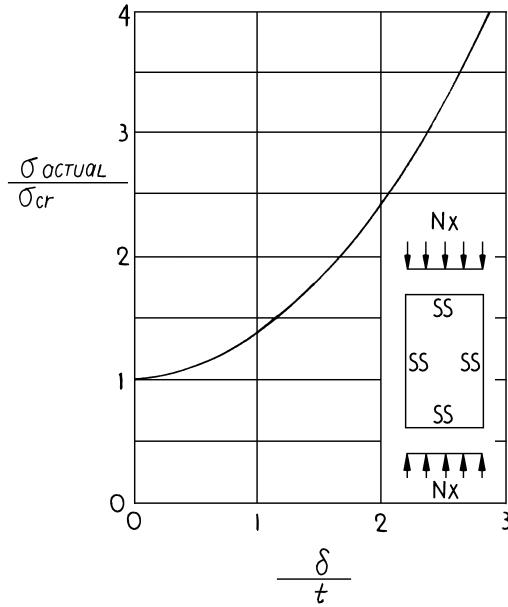


Figure 6-7. Stress vs. deflection of a rectangular plate.

while Koiter (Koiter 1943) had a different expression in the form of

$$b_e = b[1.2(\sigma_{cr}/\sigma_u)^{0.4} - 0.65(\sigma_{cr}/\sigma_u)^{0.8} + 0.45(\sigma_{cr}/\sigma_u)^{1.2}] \quad (6-26)$$

The ultimate load, P_u , carried by the plate can then be approximated by

$$P_u = \sigma_u b_e t \quad (6-27)$$

Problems

6-4 Derive Eq. (6-15).

6-5 What is the required thickness of the plate in Fig. 6-5 if $a = 40$ inches, $b = 15$ inches, $N_x = 400$ lb/inch, $E = 16,000$ ksi, and $\mu = 0.33$? Use a factor of safety of 4.0. Use increments of $1/16$ inch for thickness.

6-3 Rectangular Plates with Various Boundary Conditions

Figure 6-9 shows a plate simply supported on sides $x = 0$ and $x = a$ and subjected to force N_x . The differential Eq. (6-18) becomes

$$\nabla^4 w = -\frac{N_x}{D} \frac{\partial^2 w}{\partial x^2}. \quad (6-28)$$

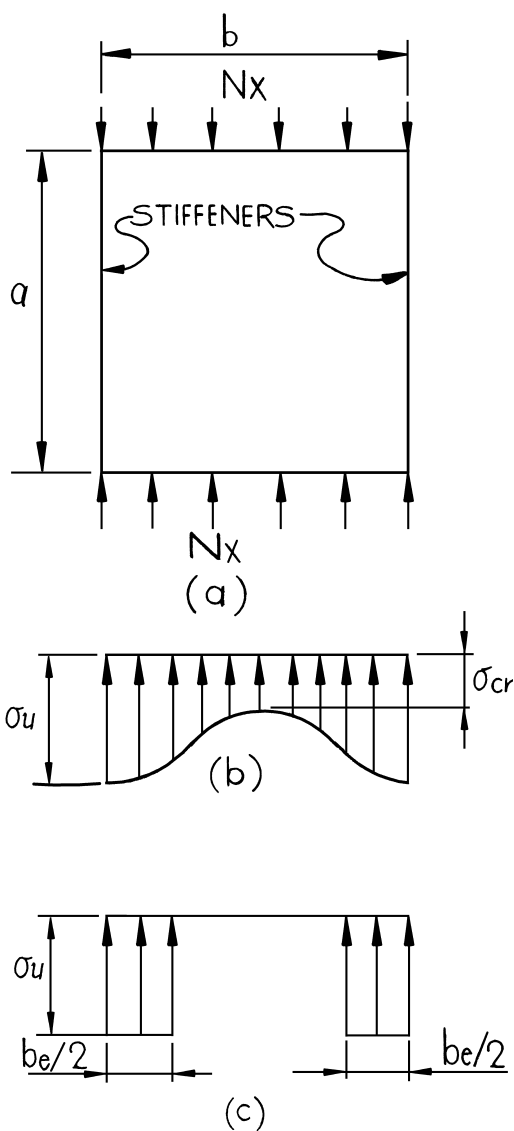


Figure 6-8. Stress distribution across plate.

Let the solution be of the form

$$w = \sum_{m=1}^{\infty} f(y) \sin \frac{m\pi x}{a}. \quad (6-29)$$

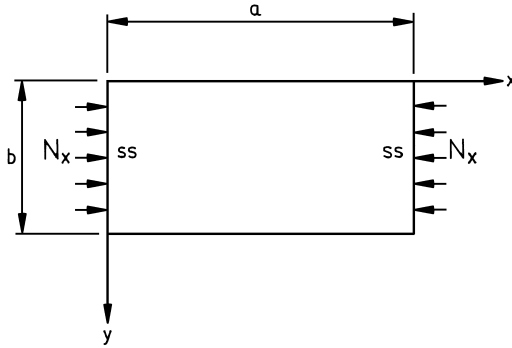


Figure 6-9. Rectangular plate with two sides simply supported.

This solution satisfies the two boundary conditions $w = M_x = 0$ at $x = 0$ and $x = a$.

Substituting Eq. (6-29) into Eq. (6-28) results in

$$\frac{d^4 f}{dy^4} - A \frac{d^2 f}{dy^2} + Bf = 0 \quad (6-30)$$

where

$$A = \frac{2m^2\pi^2}{a^2}$$

$$B = \frac{m^4\pi^4}{a^4} - \frac{N_x}{D} \frac{m^2\pi^2}{a^2}.$$

The solution of Eq. (6-30) is

$$f(y) = C_1 e^{-\alpha y} + C_2 e^{\alpha y} + C_3 \cos \beta y + C_4 \sin \beta y \quad (6-31)$$

where

$$\alpha = \sqrt{\frac{m^2\pi^2}{a^2} + \sqrt{\frac{N_x}{D} \frac{m^2\pi^2}{a^2}}}$$

$$\beta = \sqrt{-\frac{m^2\pi^2}{a^2} + \sqrt{\frac{N_x}{D} \frac{m^2\pi^2}{a^2}}}$$

Values of the constants C_1 through C_4 are obtained from the boundary conditions $y = 0$ and $y = b$.

Case 1

Side $y = 0$ is fixed and side $y = b$ is free. The four boundary conditions are

for $y = 0$, deflection $w = 0$

$$\text{rotation } \frac{\partial w}{\partial y} = 0$$

$$\text{for } y = b, \text{ moment } M_y = 0 = \frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}$$

$$\text{shear } Q = 0 = \frac{\partial^3 w}{\partial y^3} + (2 - \mu) \frac{\partial^3 w}{\partial x \partial y^2}.$$

From the first boundary condition we get

$$C_1 + C_2 + C_3 = 0$$

and from the second boundary condition we get

$$-\alpha C_1 + \alpha C_2 + \beta C_3 = 0$$

or,

$$C_1 = -\frac{C_3}{2} + \frac{\beta C_4}{2\alpha}$$

and

$$C_2 = -\frac{C_3}{2} - \frac{\beta C_4}{2\alpha}.$$

Substituting C_1 and C_2 into Eq. (6-31) gives

$$f(y) = C_3(\cos \beta y - \cosh \alpha y) + C_4\left(\sin \beta y - \frac{\beta}{\alpha} \sinh \alpha y\right)$$

With this expression and Eq. (6-29), we can solve the last two boundary conditions. This results in two simultaneous equations. The critical value of the compressive force, N_x , is determined by equating the determinant of these equations to zero. This gives

$$2gh(g^2 + h^2) \cos \beta b \cosh \alpha b = \frac{1}{\alpha\beta}(\alpha^2 h^2 - \beta^2 g^2) \sin \beta b \sinh \alpha b \quad (6-32)$$

where

$$g = \alpha^2 - \mu \frac{m^2 \pi^2}{a^2}$$

and

$$h = \beta^2 + \mu \frac{m^2 \pi^2}{a^2}.$$

For $m = 1$, the minimum value of Eq. (6-32) can be expressed in terms of stress as

$$\sigma_{cr} = \frac{\pi^2 E}{12(1 - \mu^2)(b/t)^2} K \quad (6-33)$$

where, for $\mu = 0.25$,

$$K_{\min} = 1.328. \quad (6-34)$$

Case 2

Side $y = 0$ is simply supported and side $y = b$ is free. Again starting with Eq. (6-27) and satisfying the boundary conditions at $y = 0$ and $y = b$, we get a solution (Timoshenko and Gere 1961) identical to Eq. (6-33) with K given by

$$K = 0.456 + \frac{b^2}{a^2} \text{ for } \mu = 0.25. \quad (6-35)$$

Case 3

Sides $y = 0$ and $y = b$ are fixed. In this case, the minimum value of K in Eq. (6-33) is

$$K = 7.0 \text{ for } \mu = 0.25. \quad (6-36)$$

Example 6-3

Let the plate in Fig. 6-5 be simply supported at $x = 0$ and $x = a$, simply supported at $y = 0$, and free at $y = b$. Calculate the required thickness if $a = 22$ inches, $b = 17$ inches, $N_x = 300$ lb/in, factor of safety ($F.S.$) = 2.0, $\mu = 0.25$, $\sigma_y = 36$ ksi, and $E = 29,000$ ksi.

Solution

Assume $t = 0.25$ inch. From Eq. (6-35) the minimum value of $K = 0.456$ and Eq. (6-33) becomes

$$\sigma_{cr} = \frac{\pi^2(29,000,000)}{12(0.9375)(17/0.25)^2} 0.456$$

$$= 2510 \text{ psi which is less than the yield stress.}$$

Allowable stress = $2510/2 = 1255$ psi. Actual stress = $300/0.25 = 1200$ psi. o.k.

Problems

6-6 Derive Eq. (6-30).

6-7 Derive Eq. (6-31).

6-8 Derive Eq. (6-32).

6-9 Determine the actual expression of K in Eq. (6-34). Plot K versus a/b for m values of 1, 2, and 3 and a/b values from 1.0 to 5.0.

6-4 Finite Difference Equations for Buckling

The finite difference method is a powerful tool for determining the buckling of plates with irregular shapes, complicated boundary conditions, and non-uniform applied loads. The basic differential equation of the buckling of a rectangular plate as given by Eq. (6-18) is

$$\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D} \left(p + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \quad (6-37)$$

where, N_x , N_y , and N_{xy} are entered as negative values. This equation can be transformed to a finite difference form by substituting Eqs. (5-14), (5-18), and (5-20) into the left hand side of this equation. Equations (5-12), (5-16), and (5-19) are substituted in the right hand side of Eq. (6-37). The resultant equation, after deleting the expression p , becomes

$$\begin{aligned} (1/\lambda^4) [20 w_o - 8(w_E + w_W + w_N + w_S) + 2(w_{NE} + w_{SE} + w_{SW} + w_{NW}) \\ + (w_{NN} + w_{SS} + w_{EE} + w_{WW})] = -(1/D\lambda^2) [(w_E - 2w_o + w_W) N_x \\ + (w_S - 2w_o + w_N) N_y + (1/2)(w_{NE} - w_{SE} - w_{NW} + w_{SW}) N_{xy}] \end{aligned} \quad (6-38)$$

The following example illustrates the application of this equation in determining the buckling strength of a plate.

Example 6-4

Find the buckling strength of a simply supported square plate, Fig. 6-10a, due to applied in-plane force $-N_x$. Let $N_y = N_{xy} = 0$. Assume a) $\lambda = 10$ inches and, b) $\lambda = 5$ inches.

Solution

a) The grid pattern for $\lambda = 10$ inches is shown in Fig. 6-10b. By inspection, $m = n = 1$. Applying Eq. (6-38) at node o gives

$$\begin{aligned} (1/\lambda)^4 [20 w_o - 8(0) + 2(0) + (-w_o - w_o - w_o - w_o)] \\ = (1/D)(1/\lambda^2)(-2 w_o)(-N_x) \end{aligned}$$

or,

$$N_x = 8D/\lambda^2$$

Substituting $\lambda = a/2$ in the above expression gives the critical buckling value

$$(N_x)_{cr} = 32D/a^2$$

The theoretical critical value is given by

$$(N_x)_{cr} = 4\pi^2 D/a^2 \quad \text{which is about 19\% larger than the calculated value.}$$

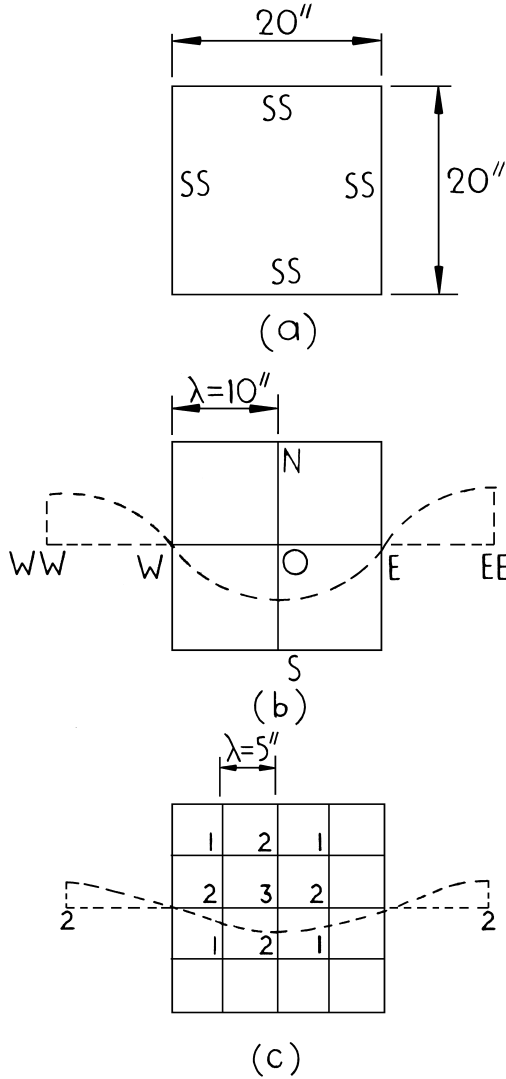


Figure 6-10. Finite difference layout of a square plate.

b) The grid pattern for $\lambda = 5$ inches is shown in Fig. 6-10c. Applying Eq. (6-38) at nodes 3, 2, and 1 gives

$$(1/\lambda)^4 [20 w_3 - 8(4 w_2) + 2(4 w_1) + (0)] = (1/D)(1/\lambda^2)(2 w_2 - 2 w_3)(-N_x)$$

$$(1/\lambda)^4 [20 w_2 - 8(w_3 + 2 w_1) + 2(2 w_2) + (w_2 - w_2)] = (1/D)(1/\lambda^2)(w_3 - 2 w_2)(-N_x)$$

$$(1/\lambda)^4 [20 w_1 - 8(2 w_2) + 2(w_3) + (w_1 + w_1 - w_1 - w_1)] = (1/D)(1/\lambda^2)(w_2 - 2 w_1)(-N_x)$$

These three equations can be written as

$$\begin{bmatrix} 20 - 2K & -32 + 2K & 8 \\ -8 + K & 24 - 2K & -16 \\ 2 & -16 + K & 20 - 2K \end{bmatrix} \begin{bmatrix} w_3 \\ w_2 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where,

$$K = \lambda^2 N_x / D$$

A non-trivial solution of these three equations is obtained by setting the determinant to zero. This gives

$$K^3 - 24K^2 + 160K - 256 = 0$$

The roots of this cubic equation are $K = 13.6569$, 8.000 , and -2.3431 . Taking the smallest root and substituting it in the expression for K gives

$$(N_x)_{cr} = 2.3431 D / \lambda^2$$

or,

$$(N_x)_{cr} = 37.49 D / a^2.$$

This value is more accurate than that obtained in part a) and is about 5% smaller than the theoretical value.

For rectangular plates with boundary conditions other than simply supported as well as other rectangular plates, a finer mesh is usually needed in order to improve the accuracy. Also, various values of m and n must be investigated in order to obtain minimum buckling load.

6-5 Other Aspects of Buckling

Temperature gradients are often encountered in many plate structures such as boiler casings, and aircraft wings. The plates in these structures, Fig. 6-11, are assumed to be continuous in the x -direction. They are also assumed to have no thermal gradients in that direction. The plates are also assumed to be supported by stiffeners at $y = 0$ and $y = b$. The stiffeners act as a heat sink resulting in a temperature gradient that varies in the y -direction in accordance with the equation

$$T = T_0 - T_1 \cos(2\pi y/b) \quad (6-39)$$

Where, T_0 , and T_1 are constants that define the temperature variation along the y -axis. It is assumed, Boley (Boley and Weiner, 1997) that the actual stress in the y -direction is zero and the actual stress in the x -direction is given by

$$\sigma_x = \alpha E T_1 \cos(2\pi y/b) \quad (6-40)$$

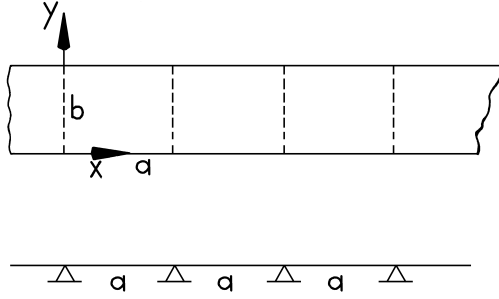


Figure 6-11. Continuous plate.

where,

E = modulus of elasticity

α = coefficient of thermal expansion.

Substituting Eq. (6-40) into Eq. (6-37) and solving the resultant equation for buckling results in

$$\sigma_{cr} = k \frac{\pi^2 E}{12(1 - \mu^2)} (t/b)^2 \quad (6-41)$$

where,

$$k = (1/2)(M_1)^2 \left\{ [(M^2 + 9)^4 + 4(M^2 + 1)^2(M^2 + 9)^2]^{1/2} - (M^2 + 9)^2 \right\} \quad (6-42)$$

and

$$M = mb/a$$

$$M_1 = a/mb$$

$$m = 1, 2, \dots$$

Equation (6-42) is shown in Fig. 6-12. A minimum value of k can be taken as $k = 3.848$.

The critical temperature T_{1cr} is defined as

$$\frac{\alpha E}{2} T_{1cr} = \sigma_{cr} \quad (6-43)$$

Substituting this expression into Eq. (6-41) gives

$$T_{1cr} = k \frac{\pi^2}{6\alpha(1 - \mu^2)} (t/b)^2 \quad (6-44)$$

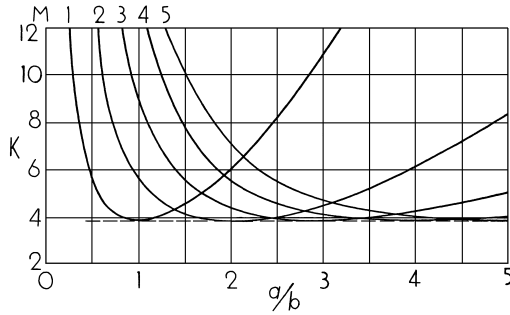


Figure 6-12. Plot of Eq. (6-42).

Example 6-5

A rectangular plate with lengths $a = 60$ inches and $b = 20$ inches has a temperature distribution in accordance with Eq. (6-39). It has a temperature of 800°F at $y = 0$ and $y = 20$ inches. The temperature at $y = 10$ inches is 1000°F . Let $E = 30 \times 10^6$ ksi, $\alpha = 6.5 \times 10^{-6}$ in/in/ $^\circ\text{F}$, yield stress = 30 ksi, and $\mu = 0.30$. What is the required thickness due to this temperature distribution assuming a factor of safety of 1.3?

Solution

From Eq. (6-39) at $y = 0$,

$$800 = T_o - T_1 \quad (1)$$

and at $y = b/2$,

$$1000 = T_o + T_1 \quad (2)$$

Solving Equations (1) and (2) gives $T_o = 900^\circ\text{F}$ and $T_1 = 100^\circ\text{F}$. The actual maximum thermal stress in the plate is obtained from Eq. (6-40) as

$$\begin{aligned} \sigma &= (6.5 \times 10^{-6})(30 \times 10^6)(100) \\ &= 19,500 \text{ psi.} \end{aligned}$$

The critical buckling stress can be taken as this number multiplied by the factor of safety of 1.3.

$$\sigma_{cr} = 19,500 \times 1.3 = 25,350 \text{ psi.}$$

Since this value is lower than the yield stress of the material, buckling controls. Substituting this value into the buckling Eq. (6-41) gives

$$25,350 = (3.848) \frac{\pi^2(30 \times 10^6)}{12(1 - 0.3^2)} (t/20)^2.$$

solving for thickness results in

$$t = 0.31 \text{ inch.}$$

6-6 Application of Buckling Expressions to Design Problems

Many codes utilize the expressions of Section 6-3 to establish buckling criteria for various members. The American Institute of Steel Construction Manual (AISC 1991) assumes the buckling stress of unsupported members in compression not to exceed the yield stress of the material. Thus, Eq. 6-33 can be written as

$$\sigma_y = \frac{\pi^2 E}{12(1 - \mu^2)(b/t)^2} K \quad (6-45)$$

or, for steel members with $\mu = 0.3$ and $E = 29,000$ ksi

$$\frac{b}{t} = 162 \sqrt{\frac{K}{\sigma_y}}. \quad (6-46)$$

Equation 6-45 is based on the assumption that the interaction between the buckling stress and the yield stress is designated by points *ABC* in Fig. 6-13. However, due to residual stress in structural members due to forming, the actual interaction curve is given by points *ADC* and Eq. 6-46 is modified by a factor of 0.7 as

$$\frac{b}{t} = 114 \sqrt{\frac{K}{\sigma_y}}. \quad (6-47)$$

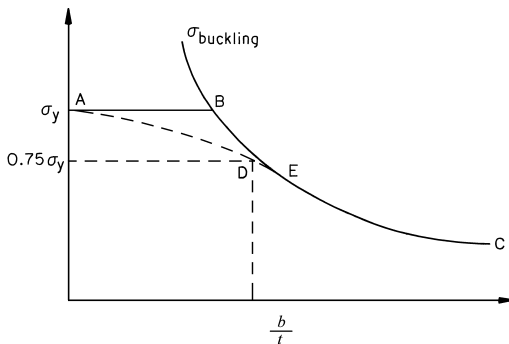


Figure 6-13. Elastic-plastic buckling.

Single-Angle Struts

Leg b of the angle shown in Fig. 6-14a is assumed free at point B . Point A is assumed simply supported because it can rotate due to deflections. Thus, Eq. 6-35 is applicable with a minimum value of $K = 0.456$ and Eq. 6-46 becomes

$$\frac{b}{t} = \frac{76}{\sqrt{\sigma_y}}.$$

Double Angles

Due to symmetry (Fig. 6-14b) the possibility of rotation is substantially reduced from that of case 1. Thus, the AISC uses the average of the simply supported-free case, Eq. 6-35, and the average of the simply supported-free and fixed-free, Eq. 6-34, cases.

$$K = \frac{0.456 + \frac{0.456 + 1.328}{2}}{2} = 0.674$$

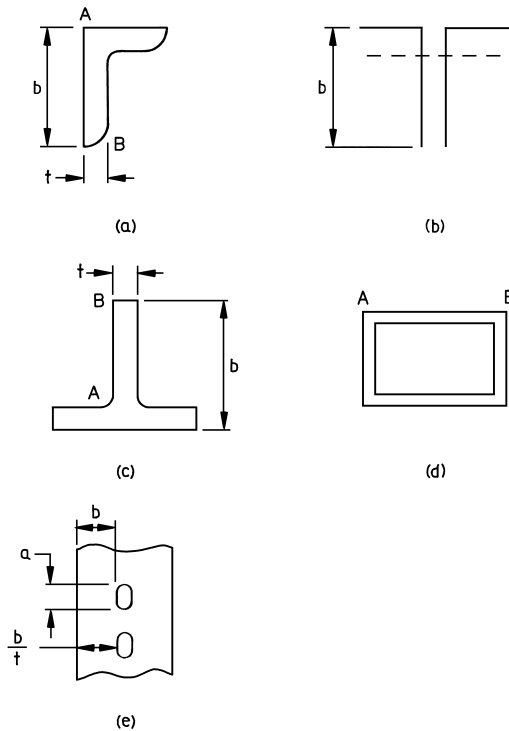


Figure 6-14. Various structural members.

and Eq. 6-47 is approximated by AISC as

$$\frac{b}{t} = \frac{95}{\sqrt{\sigma_y}}.$$

Stems of T's

For this case, point *A* in Fig. 6-14c is assumed fixed due to the much thicker flange, and point *B* is taken as free. The *K* value is taken from Eq. 6-34 and expression 6-47 gives

$$\frac{b}{t} = \frac{132}{\sqrt{\sigma_y}}.$$

The AISC reduces this value further down to

$$\frac{b}{t} = \frac{127}{\sqrt{\sigma_y}}.$$

Flanges of Box Sections

Points *A* and *B* in Fig. 6-14d are conservatively taken as simply supported. In this case, *k* = 4.0 and Eq. 6-47 becomes

$$\frac{b}{t} = \frac{228}{\sqrt{\sigma_y}}.$$

The AISC increases this value to match experimental data and it becomes

$$\frac{b}{t} = \frac{238}{\sqrt{\sigma_y}}.$$

Perforated Cover Plates

In Fig. 6-14e, the plate between the perforation and edge is assumed fixed because additional rigidity is obtained from the continuous areas between the perforations. The dimension of the perforated plate is assumed to be *a/b* = 1.0. This ratio results in a *K* value of about 7.69. This is higher than that given by Eq. 6-36 which is based on the smallest possible value.

Equation 6-47 becomes

$$\frac{b}{t} = \sqrt{\frac{317}{\sigma_y}}.$$

Other Compressed Members

Other members are assumed to have a *K* value that varies between 4.0 for simply supported edges and 7.0 for fixed edges. The AISC uses a value of *K* = 4.90. This gives

$$\frac{b}{t} = \sqrt{\frac{253}{\sigma_y}}.$$

Another standard that uses the plate buckling equations to set a criterion is the American Association of State Highway and Transportation Officials (AASHTO 1992). The equations are very similar to those of AISC. AASHTO uses the square root of the actual compressive stress in the b/t equations rather than the square root of the yield stress. Also, limits are set on the maximum b/t values for various strength steels.

A theoretical solution of the buckling of rectangular plates due to various loading and boundary conditions is available in numerous references. Two such references are Timoshenko and Gere (1961) and Iyengar (1988). Timoshenko discusses mainly isotropic plates whereas Iyengar handles composite plates.

Various NASA publications are also available for the solution of the buckling of rectangular plates with various loading and boundary conditions. NASA's Handbook of Structural Stability consists of five parts and contains numerous theoretical background and design aids. Part I, edited by Gerard and Becker, includes various classical buckling solutions for flat plates. Part II, edited by Becker, is for buckling of composite elements. Part III is for buckling of curved plates and shells and Part IV discusses failure of plates and composite elements. Parts III and IV were edited by Gerard and Becker. Compressive strength of flat stiffened panels is given in Part V which is edited by Gerard.

Vibration of Plates

7-1 Introduction

Many structural parts such as aircraft components, boiler panels, industrial ducts, and machine parts are subjected to loading conditions that are conducive to vibration since they are in close proximity to reciprocating equipment or high fluid velocity. The mechanism of such plate vibration has been studied extensively in the last few decades. Numerous books and articles have been written dealing extensively with this topic. Leissa's book (Leissa 1969) gives solutions to extensive variety of plate geometries, boundary conditions, and material properties and serves as an excellent overall reference book. Another book by Gorman (Gorman 1982) develops detailed equations and solutions to a wide variety of rectangular plates. Other books on plate vibrations are also referenced in subsequent paragraphs.

In this chapter a brief description of the theory of the vibration of plates is presented for different plate configurations and boundary conditions to illustrate the general procedure for obtaining natural frequencies. The reader is encouraged to review the referenced books in this chapter for more in-depth treatment of the plate vibration problem.

7-2 General Equations for Rectangular Plates

The basic differential equation for the bending of a thin rectangular plate given in Chapter 1 is

$$\nabla^4 w = p/D \quad (7-1)$$

where, w is the deflection of the plate and is taken as positive downwards, p is the surface load, and D is the stiffness. In a free vibrating plate, the applied surface loads are disregarded. However, the plate mass, Fig. 7-1, exerts an inertial force during vibration. The inertial force is expressed as $\rho \partial^2 w / \partial \tau^2$ where ρ is the mass density per unit area, $\text{lbs-sec}^2 / \text{in}^2$, of the plate and τ is time. The value of ρ

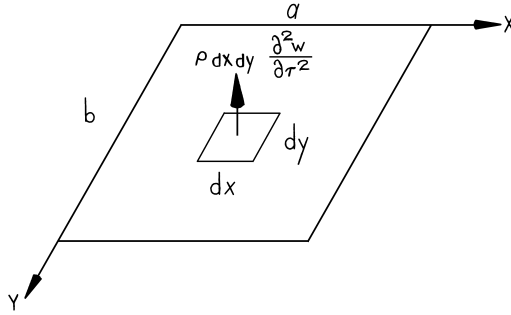


Figure 7-1. Inertial force in a rectangular plate.

is equal to the quantity (density of the plate/acceleration) (thickness). Substituting these values into Eq. (7-1) gives

$$\nabla^4 w = -(\rho/D) \partial^2 w / \partial \tau^2 \quad (7-2)$$

where, the negative sign indicated that the inertial force acts in the opposite direction to the deflection. The deflection, w , in this equation is a function of the coordinates x and y as well as time τ . Equation (7-2) can be written as

$$\frac{\partial^4 w(x, y, \tau)}{\partial x^4} + 2 \frac{\partial^4 w(x, y, \tau)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y, \tau)}{\partial y^4} = -\frac{\rho}{D} \frac{\partial^2 w(x, y, \tau)}{\partial \tau^2} \quad (7-3)$$

The solution of this equation can be simplified if the deflection w is expressed as a function of two separate quantities. The first quantity is expressed in terms of x and y and the second in terms of time. Thus,

$$w(x, y, \tau) = w(x, y) T(\tau) \quad (7-4)$$

Substituting this expression into Eq. (7-3) and re-arranging terms results in

$$\begin{aligned} \frac{D}{\rho} \left(\frac{\partial^4 w(x, y)}{\partial x^4} + 2 \frac{\partial^4 w(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y)}{\partial y^4} \right) / w(x, y) \\ = -\frac{d^2 T(\tau)}{d\tau^2} / T(\tau) \end{aligned} \quad (7-5)$$

This equation is easier to solve than Eq. (7-3) since the left hand side of this equation is a function of dimensions x and y only while the right hand side is a function of time. A non-trivial solution of Eq. (7-5) is obtained when the left-hand and the right hand sides are set equal to a constant, ω^2 . The right hand side of Eq. (7-5) becomes

$$\frac{d^2 T(\tau)}{d\tau^2} + \omega^2 T(\tau) = 0 \quad (7-6)$$

The solution of this second order differential equation is

$$T(\tau) = A \sin \omega \tau + B \cos \omega \tau \quad (7-7)$$

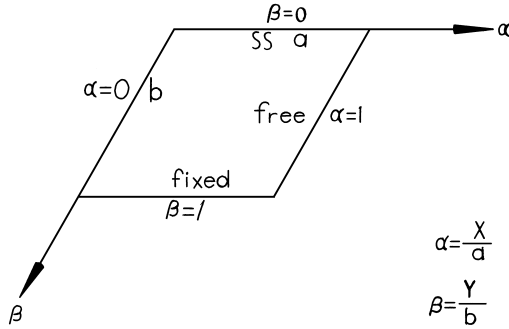


Figure 7-2. α and β designation in a rectangular plate.

Where, A and B are constants. The quantity $T(\tau)$ varies sinusoidally with time and the quantity ω defines the natural frequency of the plate in radians per second. Since the natural frequency may be determined at any time in the cycle, we can set τ equal to zero in Eq. (7-7) and get $B = 0$. Equation (7-7) becomes

$$T(\tau) = A \sin \omega \tau \quad (7-8)$$

The right hand side of Eq. (7-5), when equated to ω^2 and rearranged, becomes

$$\frac{\partial^4 w(x, y)}{\partial x^4} + 2 \frac{\partial^4 w(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 w(x, y)}{\partial y^4} - \frac{\omega^2 \rho}{D} w(x, y) = 0 \quad (7-9)$$

This equation includes the geometric property x, y of the plate, the stiffness D , and the natural frequency ω . The solution of this equation can be simplified (Gormann 1982) to a non-dimensional form by defining, Fig. 7-2, $\alpha = x/a$, $\beta = y/b$, and $\gamma = b/a$. Substituting these quantities into Eq. (7-9) and rearranging various terms results in

$$\frac{\partial^4 w(\alpha, \beta)}{\partial \alpha^4} + 2\gamma^2 \frac{\partial^4 w(\alpha, \beta)}{\partial \alpha^2 \partial \beta^2} + \gamma^4 \frac{\partial^4 w(x, y)}{\partial \beta^4} - \gamma^4 \lambda^4 w(x, y) = 0 \quad (7-10)$$

where,

$$\lambda^2 = \omega a^2 (\rho/D)^{1/2} \quad (7-11)$$

This differential equation is dimensionless and it expresses the free vibration of a rectangular plate. The boundary conditions associated with this non-dimensional equation are

Fixed edge

Equations (1-29) and (1-30) for fixed edge $y = b$ in Fig. 1-9 can be re-written in non-dimensional terms in accordance with Fig. 7-2 as

$$w(\alpha, \beta) = 0 \quad (7-12)$$

and

$$\frac{\partial w(\alpha, \beta)}{\partial \beta} = 0 \quad (7-13)$$

Simply supported edge

Similarly, Eqs. (1-31) and (1-33) for simply supported edge $y = 0$, Fig. 1-9, are expressed in the terminology of Fig. 7-2 as

$$w(\alpha, \beta) = 0 \quad (7-14)$$

and

$$\frac{\partial^2 w(\alpha, \beta)}{\partial \beta^2} = 0 \quad (7-15)$$

Free edge

For a free edge, Eqs. (1-34) and (1-35) become

$$\frac{\partial^2 w(\alpha, \beta)}{\partial \alpha^2} + \frac{\mu}{\lambda^2} \frac{\partial^2 w(\alpha, \beta)}{\partial \beta^2} = 0 \quad (7-16)$$

and

$$\frac{\partial^3 w(\alpha, \beta)}{\partial \alpha^3} + \frac{2 - \mu}{\lambda^2} \frac{\partial^3 w(\alpha, \beta)}{\partial \alpha \partial \beta^2} = 0 \quad (7-17)$$

Solution of the differential equation (7-10) can best be achieved by using the Levy approach, Eq. (1-43). This equation is written as

$$w(\alpha, \beta) = \sum_{m=1}^{\infty} f_m(\beta) \sin m\pi\alpha \quad (7-18)$$

Substituting this expression into Eq. (7-10) results in an expression similar to Eq. (1-44) as

$$\frac{d^4 f_m(\beta)}{d\beta^4} - 2\gamma^2(m\pi)^2 \frac{d^2 f_m(\beta)}{d\beta^2} + \gamma^4[(m\pi)^4 - \lambda^4] f_m(\beta) = 0 \quad (7-19)$$

A non-trivial solution of this fourth order differential equation is:

For $\lambda^2 > (m\pi)^2$

$$f_m(\beta) = A_m \cosh \eta_m \beta + B_m \sinh \eta_m \beta + C_m \sin \xi_m \beta + D_m \cos \xi_m \beta \quad (7-20)$$

and for $\lambda^2 < (m\pi)^2$

$$f_m(\beta) = A_m \cosh \eta_m \beta + B_m \sinh \eta_m \beta + C_m \sinh \xi_m \beta + D_m \cosh \xi_m \beta \quad (7-21)$$

where,

A_m , B_m , C_m , and D_m are constants obtained from boundary conditions. The expressions η_m and ξ_m are defined as

$$\eta_m = \gamma [\lambda^2 + (m\pi)^2]^{1/2} \quad (7-22)$$

and

$$\xi_m = \gamma [\lambda^2 - (m\pi)^2]^{1/2} \text{ or } \xi_m = \gamma [(m\pi)^2 - \lambda^2]^{1/2} \text{ whichever is real. } (7-23)$$

Thus, the natural frequency of a rectangular plate with various boundary conditions is determined from Eqs. (7-19), (7-20), and (7-21).

7-3 Simply Supported Rectangular Plates

The governing equation for the vibration of a simply supported plate is obtained from Eq. (7-19). Its solution is given by Eqs. (7-20) and (7-21). We place the coordinate system as shown in Fig. 7-3 to simplify the derivations. The deflection pattern around the α axis will be solved for either the symmetrical or non-symmetrical shape.

Symmetrical Condition

For the symmetrical condition, constants B_m and C_m must be set to zero and Eqs. (7-20) and (7-21) become

For $\lambda^2 > (m\pi)^2$

$$f_m(b) = A_m \cosh \eta_m \beta + D_m \cos \xi_m \beta \quad (7-24)$$

and for $\lambda^2 < (m\pi)^2$

$$f_m(b) = A_m \cosh \eta_m \beta + D_m \cosh \xi_m \beta \quad (7-25)$$

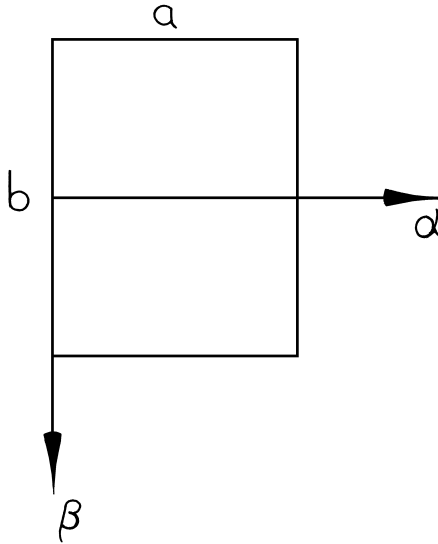


Figure 7-3. Simply supported plate.

Equation (7-18) already satisfies the simply supported boundary conditions at $\alpha = 0$ and $\alpha = 1$. The boundary conditions at $b = +\frac{1}{2}$ are given by Eqs. (7-12) and (7-13). Substituting Eq. (7-18) into these two boundary conditions and using Eqs. (7-24) and (7-25) yields

For $\lambda^2 > (m\pi)^2$

$$A_m \cosh \frac{1}{2} \eta_m + D_m \cos \frac{1}{2} \xi_m = 0 \quad (7-26)$$

$$A_m \eta_m^2 \cosh \frac{1}{2} \eta_m + D_m \xi_m^2 \cos \frac{1}{2} \xi_m = 0 \quad (7-27)$$

and for $\lambda^2 < (m\pi)^2$

$$A_m \cosh \frac{1}{2} \eta_m + D_m \cos \frac{1}{2} \xi_m = 0 \quad (7-28)$$

$$A_m \eta_m^2 \cosh \frac{1}{2} \eta_m - D_m \xi_m^2 \cos \frac{1}{2} \xi_m = 0 \quad (7-29)$$

Equations (7-26) and (7-27), or Eqs. (7-28) and (7-29), can be expressed in matrix form as

$$\begin{bmatrix} \cosh \frac{1}{2} \eta_m & \cos \frac{1}{2} \xi_m \\ \eta_m^2 \cosh \frac{1}{2} \eta_m & -\xi_m^2 \cos \frac{1}{2} \xi_m \end{bmatrix} \begin{bmatrix} A_m \\ D_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (7-30)$$

A non-trivial solution of Eq. (7-30) is obtained when the determinate of the coefficient matrix is set to zero (Hartog, 1984). This gives an eigenvalue equation of the form

$$(\eta_m^2 + \xi_m^2) \cosh \frac{1}{2} \eta_m \cos \frac{1}{2} \xi_m = 0 \quad (7-31)$$

where, $\xi_m = \pi, 3\pi, 5\pi, \dots$

Substituting for ξ_m odd multiples of π , Eq. (7-23) becomes

$$\lambda^2 = (m\pi)^2 + (n\pi)^2/\gamma^2 \quad (7-32)$$

where, n is odd. Combining Eqs. (7-11) and (7-32) gives the required natural frequency of a simply supported plate due to symmetric deflection around the center line of the plate

$$\omega = (1/a^2)(D/\rho)^{1/2}[(m\pi)^2 + (n\pi)^2/\gamma^2] \quad (7-33)$$

where, $n = 1, 3, 5, \dots$

Non-symmetrical Condition

For an non-symmetrical deflection, w , the constants A_m and D_m in Eqs. (7-20) and (7-21) vanish. Setting the determinate of the matrix coefficients to zero results in an eigenvalue expression similar to that in Eq. (7-31) of the form

$$(\eta_m^2 + \xi_m^2) \sinh \frac{1}{2} \eta_m \sin \frac{1}{2} \xi_m = 0 \quad (7-34)$$

Table 7-1. λ^2 values for simply supported plates

m	n	a/b		
		1	2	3
1	1	19.74	49.35	98.70
1	2	49.35	167.78	365.18
1	3	98.70	365.18	809.31
2	1	49.35	78.96	128.30
2	2	78.96	197.39	394.78
2	3	128.30	394.78	838.92
3	1	98.70	128.30	177.65
3	2	128.30	246.74	444.13
3	3	177.65	444.13	888.26

$$\omega = (\lambda/a)^2 (D/\rho)^{1/2}$$

where, $\xi_m = 2\pi, 4\pi, 6\pi, \dots$

Substituting for ξ_m even multiples of π , Eq. (7-23) becomes

$$\lambda^2 = (m\pi)^2 + (n\pi)^2/\gamma^2 \quad (7-35)$$

where, n is even. Again combining Eqs. (7-11) and (7-35) gives the required natural frequency of a simply supported plate due to non-symmetric deflection around the center line of the plate

$$\omega = (1/a^2)(D/\rho)^{1/2}[(m\pi)^2 + (n\pi)^2/\gamma^2] \quad (7-36)$$

where, $n = 2, 4, 6, \dots$

Numerical values of Eqs. (7-32) and (7-35) are shown in Table 7-1. The natural frequency is shown in Table 7-2 in ascending order. It is of interest to note that the natural frequency for a rectangular plate having m_i, n_j is different than that for m_j, n_i . This property is of significance when designing or stiffening a plate in close proximity to reciprocating equipment.

Example 7-1

What are the first and second natural frequencies of a simply supported rectangular plate have $a = 30$ inches, $b = 20$ inches, $t = 3/32$ inch, $E = 30,000$ ksi, $\mu = 0.3$, and $\rho = 6.88E-5$ lb-sec²/in³?

Solution:

The stiffness of the plate is given by

$$\begin{aligned} D &= (30,000,000)(0.0938)^3/12(1 - 0.3^2) \\ &= 2263.7 \end{aligned}$$

Table 7-2. λ^2 for various combinations of $m = 1, 2, 3$ and $n = 1, 2, 3$

a/b		
1	2	3
19.74	49.35	98.70
49.35	78.96	128.30
78.96	128.30	177.65
98.70	167.78	365.18
128.30	197.39	394.78
177.65	246.74	444.13
	365.18	809.31
	394.78	838.92
	444.13	888.26

$$\omega = (\lambda/a)^2 (D/\rho)^{1/2}$$

and $\gamma = b/a = 0.667$. For $m = 1$ and $n = 1$, Eq. (7-33) becomes

$$\begin{aligned}\omega &= (1/30^2)(2263.7/0.0000688)^{1/2}[(\pi)^2 + (\pi)^2/0.444] \\ &= 6.3734(32.0784) \\ &= 204.4 \text{ Hz} \\ &= 32.5 \text{ cps}\end{aligned}$$

For $m = 2$ and $n = 1$, Eq. (7-33) gives

$$\begin{aligned}\omega &= 6.3734[(2\pi)^2 + (\pi)^2/0.444] \\ &= 393.1 \text{ Hz} \\ &= 62.6 \text{ cps}\end{aligned}$$

For $m = 1$ and $n = 2$, Eq. (7-35) yields

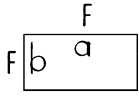
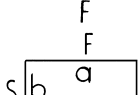
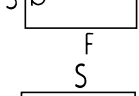
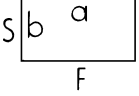
$$\begin{aligned}\omega &= 6.3734[(\pi)^2 + (2\pi)^2/0.444] \\ &= 629.6 \text{ Hz} \\ &= 100.2 \text{ cps}\end{aligned}$$

7-4 Rectangular Plates with Various Boundary Conditions

The natural frequency of rectangular plates with boundary conditions other than simply supported is obtained from Eq. (7-10) by satisfying the appropriate boundary conditions. Thus, for a plate simply supported at two opposite edges and fixed at the other two edges, the equation for the symmetric case becomes

$$\begin{bmatrix} \cosh^{1/2} \eta_{lm} & \cos^{1/2} \xi_m \\ \eta_{lm} \sinh^{1/2} \eta_{lm} & -\xi_m \sin^{1/2} \xi_m \end{bmatrix} \begin{bmatrix} A_m \\ D_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (7-37)$$

Table 7-3. λ^2 for rectangular plates with various boundary conditions

Case	a/b		
	1	2	3
	36.00	98.00	212.40
	28.95	95.28	206.91
	23.70	69.20	145.80
	31.82	96.61	207.80

F = fixed edge.

S = simply supported edge.

$$\omega = (\lambda/a)^2 (D/\rho)^{1/2}$$

This gives an eigenvalue equation of the form

$$\xi_m \cosh^{1/2} \eta_m \sin^{1/2} \xi_m + \eta_m \sinh^{1/2} \eta_m \cos^{1/2} \xi_m = 0 \quad (7-38)$$

while for the non-symmetrical case the equation becomes

$$\begin{bmatrix} \sinh^{1/2} \eta_m & \sin^{1/2} \xi_m \\ \eta_m \cosh^{1/2} \eta_m & \xi_m \cos^{1/2} \xi_m \end{bmatrix} \begin{bmatrix} B_m \\ C_m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (7-39)$$

and the eigenvalue equation is expressed as

$$\xi_m \sinh^{1/2} \eta_m \cos^{1/2} \xi_m - \eta_m \cosh^{1/2} \eta_m \sin^{1/2} \xi_m = 0 \quad (7-40)$$

The solution of Eqs. (7-38) and (7-40) cannot be obtained in a closed form. The solution is obtained by increasing the value of λ in the expressions ξ_m and η_m by small increments until the equations reverse sign from positive to negative or vice versa. This type of approach is used in obtaining the natural frequency of most rectangular plates with boundary conditions other than simply supported. Some

fundamental frequencies of plates with various boundary conditions are shown in Table 7-3.

Problem

7-1 A simply supported rectangular plate has $t = 0.1875$ inch, $a = 24$ inches and $b = 18$ inches. The plate is located close to a reciprocating equipment with an ω of 125 cps. Using a vibration factor of safety of 1.2, determine whether or not the plate is adequate for service. Let $E = 30,000$ ksi, $\mu = 0.3$, and $\rho = 6.88E-5$ lb-sec²/in³.

7-5 Circular Plates

The governing equation for the vibration of a circular plate is obtained by substituting the homogeneous deflection expression given by Eq. (3-27) in polar coordinates into the vibration differential Eq. (7-2). Assuming a solid plate without a central hole and symmetric boundary condition, the deflection becomes

$$w_n = [A_n J_n((\lambda/a)r) + B_n I_n((\lambda/a)r)] \cos n\theta \quad (7-41)$$

Where, A_n , B_n are constants and J_n , I_n are Bessel functions and

$$\lambda^2 = \omega^2 a^2 (\rho/D)^{1/2} \quad (7-42)$$

with “a” as the radius of the plate.

For a fixed plate, the boundary conditions at $r = a$ are

$$w = 0 \quad \text{and} \quad w/r = 0$$

Substituting these two boundary conditions into Eq. (7-41) results in

$$\begin{bmatrix} J_n(\lambda) & I_n(\lambda) \\ J'_n(\lambda) & I'_n(\lambda) \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Setting the determinate to zero and using the recursion relationship between various Bessel functions yields

$$J_n(\lambda)I_{n+1}(\lambda) + I_n(\lambda)J_{n+1}(\lambda) = 0 \quad (7-43)$$

The fundamental natural frequency is obtained by solving Eq. (7-43) with $n = 0$. This gives $\lambda^2 = 10.22$ and

$$\omega = (\lambda/a)^2 (D/\rho)^{1/2} \quad (7-44)$$

A similar derivation may be carried out for a simply supported plate where the deflection and moment at $r = a$ are zero. This gives $\lambda^2 = 4.98$ for $\mu = 0.30$.

The modal shapes of a vibrating circular plate for values of $n > 0$ are a function of both radius r and angle θ .

The natural frequency of circular plates with a central hole and various boundary conditions is given in many references such as Lissa (Leissa 1969) and Szilard (Szilard 1974).

Membrane Theory of Shells of Revolution

8-1 Basic Equations of Equilibrium

The membrane shell theory is used extensively in designing such structures as flat-bottom tanks, pressure vessel components (Fig. 8-1) and dome roofs. The membrane theory assumes that equilibrium in the shell is achieved by having the in-plane membrane forces resist all applied loads without any bending moments. The theory gives accurate results as long as the applied loads are distributed over a large area of the shell such as pressure and wind loads. The membrane forces by themselves cannot resist local concentrated loads. Bending moments are needed to resist such loads as discussed in Chapter 10. The basic assumptions made in deriving the membrane theory (Gibson 1965) are

1. The shell is homogeneous and isotropic.
2. The thickness of the shell is small compared to its radius of curvature.
3. The bending strains are negligible and only strains in the middle surface are considered.
4. The deflection of the shell due to applied loads is small.

In order to derive the governing equations for the membrane theory of shells, we need to define the shell geometry. The middle surface of a shell of constant thickness may be considered a surface of revolution. A surface of revolution is obtained by rotating a plane curve about an axis lying in the plane of the curve. This curve is called a meridian (Fig. 8-2). Any point in the middle surface can be described by first specifying the meridian on which it is located and second by specifying a quantity, called a parallel circle, that varies along the meridian and is constant on a circle around the axis of the shell. The meridian is defined by the angle θ and the parallel circle by ϕ as shown in Fig. 8-2.

Define r (Fig. 8-3) as the radius from the axis of rotation to any given point o on the surface; r_1 as the radius from point o to the center of curvature of the



Figure 8-1. A pressure vessel. (Courtesy of the Nooter Corp., St. Louis, MO.)

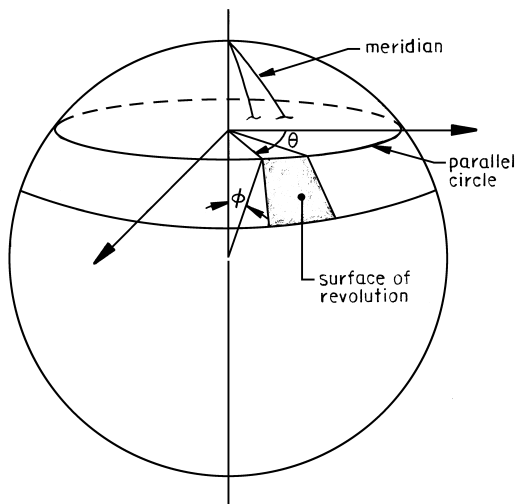


Figure 8-2. Surface of revolution.

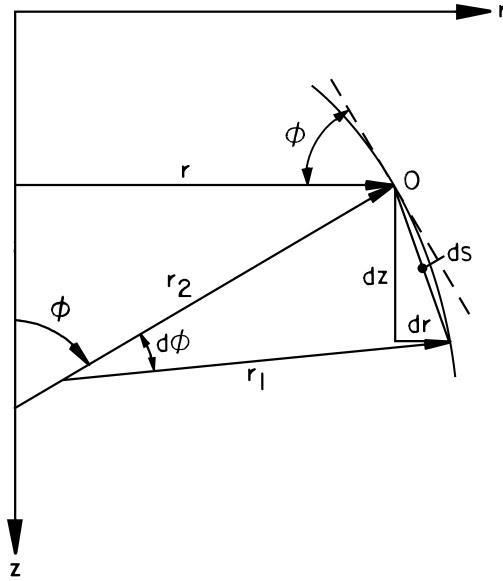


Figure 8-3. Shell geometry.

meridian; and r_2 as the radius from the axis of revolution to point o, and it is perpendicular to the meridian. Then from Fig. 8-3,

$$r = r_2 \sin \phi, \quad ds = r_1 d\phi, \quad \text{and} \quad dr = ds \cos \phi. \quad (8-1)$$

The interaction between the applied loads and resultant membrane forces is obtained from statics and is shown in Fig. 8-4. Shell forces N_ϕ and N_θ are membrane forces in the meridional and circumferential directions, respectively. Shearing forces $N_{\phi\theta}$ and $N_{\theta\phi}$ are as shown in Fig. 8-4. Applied load p_r is perpendicular to the surface of the shell; load p_ϕ is in the meridional direction; and load p_θ is in the circumferential direction. All forces are positive as shown in Fig. 8-4.

The first equation of equilibrium is obtained by summing forces parallel to the tangent at the meridian. This yields

$$\begin{aligned} N_{\theta\phi} r_1 d\phi - \left(N_{\theta\phi} + \frac{\partial N_{\theta\phi}}{\partial \phi} d\phi \right) r_1 d\phi - N_\phi r d\theta \\ + \left(N_\phi + \frac{\partial N_\phi}{\partial \phi} d\phi \right) \left(r + \frac{\partial r}{\partial \phi} d\phi \right) d\theta \\ + p_\phi r d\theta r_1 d\phi - N_\theta r_1 d\phi d\theta \cos \phi = 0. \end{aligned} \quad (8-2)$$

The last term in Eq. (8-2) is the component of N_θ that is parallel to the tangent at the meridian. It is obtained from Fig. 8-5a by finding the components F_1 and

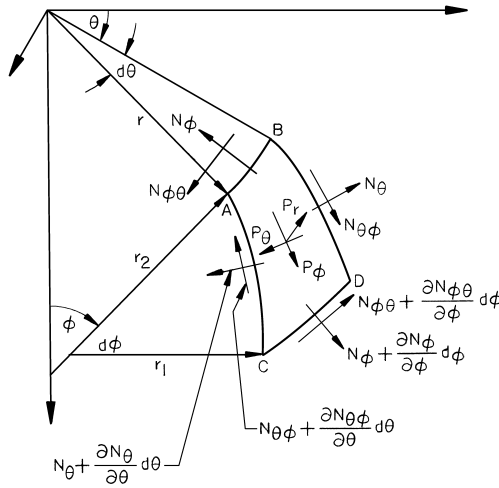


Figure 8-4. Membrane forces and applied loads.

F_2 . These are expressed as

$$F_1 + F_2 = (N_\theta r_1 d\phi) \frac{d\theta}{2} + \left(N_\theta + \frac{\partial N_\theta}{\partial \theta} d\theta \right) r_1 d\phi \frac{d\theta}{2}.$$

Neglecting terms of higher order results in

$$F_1 + F_2 = N_\theta r_1 d\phi d\theta.$$

The component of F_1 and F_2 that is parallel to the tangent at the meridian is shown in Fig. 8-5b and is given by

$$N_\theta r_1 d\phi d\theta \cos \phi.$$

This value is shown as the last expression in Eq. (8-2). Equation (8-2) can be simplified as

$$\frac{\partial}{\partial \phi} (r N_\phi) - r_1 \frac{\partial N_{\theta\phi}}{\partial \theta} - r_1 N_\theta \cos \phi + p_\phi r r_1 = 0. \quad (8-3)$$

The second equation of equilibrium is obtained from summation of forces in the direction of parallel circles. Referring to Fig. 8-4,

$$\begin{aligned} & N_{\phi\theta} r d\theta - \left(N_{\phi\theta} + \frac{\partial N_{\phi\theta}}{\partial \phi} d\phi \right) \left(r + \frac{\partial r}{\partial \phi} d\phi \right) d\theta \\ & - N_\theta r_1 d\phi + \left(N_\theta + \frac{\partial N_\theta}{\partial \theta} d\theta \right) (r_1 d\phi) \\ & + p_\theta r d\theta r_1 d\phi - N_{\theta\phi} r_1 d\phi \frac{\cos \phi d\theta}{2} \\ & - \left(N_{\theta\phi} + \frac{\partial N_{\theta\phi}}{\partial \theta} d\theta \right) (r_1 d\phi) \frac{\cos \phi d\theta}{2} = 0. \end{aligned} \quad (8-4)$$

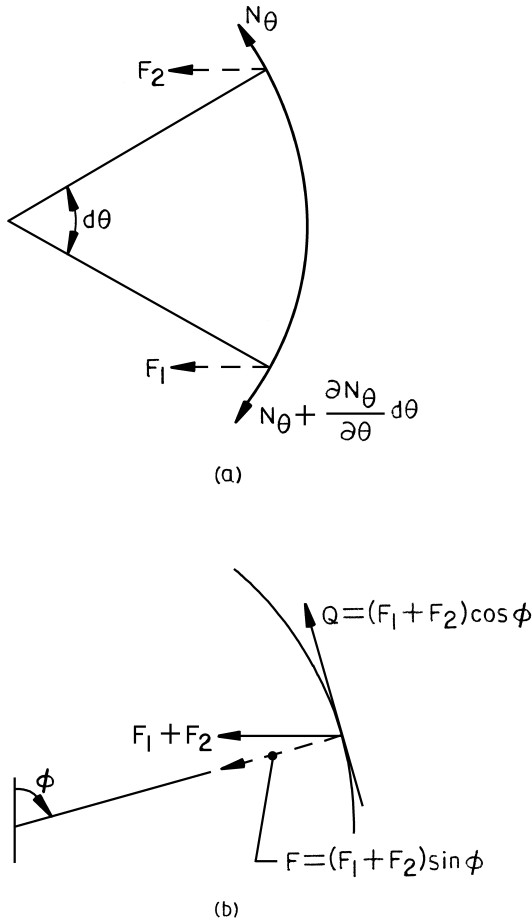


Figure 8-5. Components of N_{θ} .

The last two expressions in Eq. (8-4) are obtained from Fig. 8-6 and are the component of $N_{\theta\phi}$ in the direction of parallel circles. Hence, from Fig. 8-6a,

$$T = T_1 + T_2 = N_{\theta\phi} r_1 d\phi \frac{d\alpha}{2} + \left(N_{\theta\phi} + \frac{\partial N_{\theta\phi}}{\partial \theta} d\theta \right) r_1 d\phi \frac{d\alpha}{2}. \quad (8-5)$$

The value of $d\alpha$ can be expressed in terms of θ and ϕ as shown in Fig. 8-6b.

$$r' d\alpha = r d\theta$$

or

$$d\alpha = \frac{r d\theta}{r_2 \tan \phi}$$

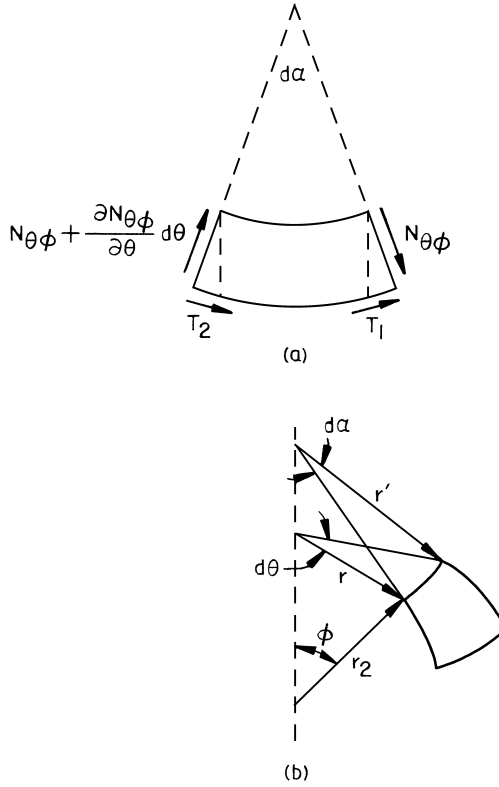


Figure 8-6. Components of $N_{\theta\phi}$.

$$= \frac{r_2 \sin \phi}{r_2 \tan \phi} d\theta$$

$$d\alpha = \cos \phi d\theta. \quad (8-6)$$

Substituting Eq. (8-6) into Eq. (8-5) results in the expression that is shown as the last term in Eq. (8-4). Equation (8-4) can now be simplified to

$$\frac{\partial}{\partial \phi} (r N_{\phi\theta}) - r_1 \frac{\partial N_{\theta}}{\partial \theta} + r_1 N_{\theta\phi} \cos \phi - p_{\theta} r r_1 = 0. \quad (8-7)$$

This is the second equation of equilibrium of the infinitesimal element shown in Fig. 8-4. The last equation of equilibrium is obtained by summing forces perpendicular to the middle surface. Referring to Figs. 8-4, 8-5, and 8-7,

$$(N_{\theta} r_1 d\phi d\theta) \sin \phi - p_r d\theta r_1 d\phi + N_{\phi} r d\theta d\phi = 0$$

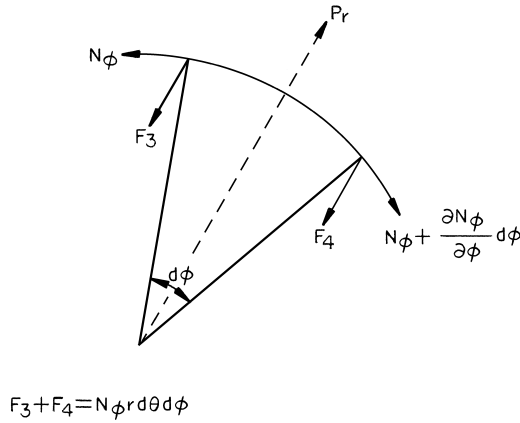


Figure 8-7. Components of N_ϕ .

or

$$N_\theta r_1 \sin \phi + N_\phi r = p_r r r_1. \quad (8-8)$$

Equations (8-3), (8-7), and (8-8) are the three equations of equilibrium of a shell of revolution subjected to axisymmetric loads.

8-2 Ellipsoidal and Spherical Shells Subjected to Axisymmetric Loads

In many structural applications, loads such as dead weight, snow, and pressure are symmetric around the axis of the shell. Hence, all forces and deformations must also be symmetric around the axis. Accordingly, all loads and forces are independent of θ and all derivatives with respect to θ are zero. Equation (8-3) reduces to

$$\frac{\partial}{\partial \phi} (r N_\phi) - r_1 N_\theta \cos \phi = -p_\phi r r_1. \quad (8-9)$$

Equation (8-7) becomes

$$\frac{\partial}{\partial \phi} (r N_{\theta\phi}) - r_1 N_{\theta\phi} \cos \phi = p_\phi r r_1. \quad (8-10)$$

In this equation, we let the cross shears $N_{\phi\theta} = N_{\theta\phi}$ in order to maintain equilibrium.

Equation (8-8) can be expressed as

$$\frac{N_\theta}{r_2} + \frac{N_\phi}{r_1} = p_r. \quad (8-11)$$

Equation (8-10) describes a torsion condition in the shell. This condition produces deformations around the axis of the shell. However, the deformation

around the axis is zero due to axisymmetric loads. Hence, we must set $N_{\theta\phi} = p_{\theta} = 0$ and we disregard Eq. (8-10) from further consideration.

Substituting Eq. (8-11) into Eq. (8-9) gives

$$N_{\phi} = \frac{1}{r_2 \sin^2 \phi} \left[\int r_1 r_2 (p_r \cos \phi - p_{\phi} \sin \phi) \sin \phi \, d\phi + C \right]. \quad (8-12)$$

The constant of integration C in Eq. (8-12) is additionally used to take into consideration the effect of any additional applied loads that cannot be defined by p_r and p_{ϕ} such as weight of contents.

Equations (8-11) and (8-12) are the two governing equations for designing double-curvature shells under membrane action.

Example 8-1

Determine the expressions for N_{ϕ} and N_{θ} due to internal pressure p in an ellipsoidal shell (Fig. 8-8) of radii a and b .

Solution

For internal pressure we define $p_r = p$ and $p_{\phi} = 0$. Then from Eqs. (8-1) and (8-11)

$$N_{\phi} = \frac{1}{r_2 \sin^2 \phi} (p \int r \, dr + C)$$

$$N_{\phi} = \frac{1}{r_2 \sin^2 \phi} \left(\frac{pr^2}{2} + C \right). \quad (1)$$

The constant C is obtained from the following boundary condition:

$$\text{At } \phi = \pi/2, \quad r_2 = r \quad \text{and} \quad N_{\phi} = pr/2.$$

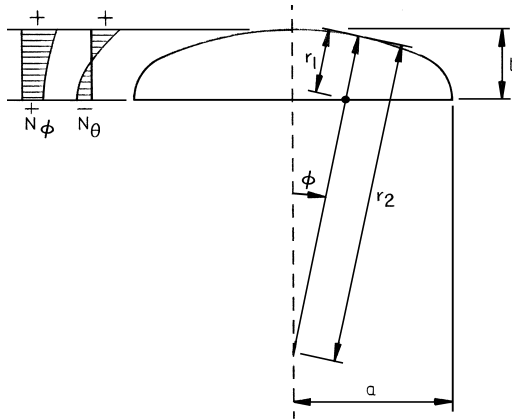


Figure 8-8. Ellipsoidal shell with internal pressure.

Hence, from Eq. (1) we get $C = 0$ and N_ϕ can be expressed as

$$N_\phi = \frac{pr^2}{2r_2 \sin^2 \phi}$$

or

$$N_\phi = pr_2/2. \quad (2)$$

From Eq. (8-11),

$$N_\theta = pr_2 \left(1 - \frac{r_2}{2r_1} \right). \quad (3)$$

From analytical geometry, the relationship between the major and minor axes of an ellipse and r_1 and r_2 is given by

$$r_1 = \frac{a^2 b^2}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{3/2}}$$

$$r_2 = \frac{a^2}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{1/2}}.$$

Substituting these expressions into Eqs. (2) and (3) gives the following expressions for membrane forces in ellipsoidal shells due to internal pressure:

$$N_\phi = \frac{pa^2}{2} \frac{1}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{1/2}} \quad (4)$$

$$N_\theta = \frac{pa^2}{2b^2} \frac{b^2 - (a^2 - b^2) \sin^2 \phi}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{1/2}}. \quad (5)$$

A plot of Eqs. (4) and (5) is shown in Fig. 8-8. Equation (4) for the longitudinal force, N_ϕ , is always in tension regardless of the a/b ratio. Equation (5) for N_θ on the other hand gives compressive circumferential, hoop, forces near the equator when the value $a/b \geq \sqrt{2}$. For large a/b ratios under internal pressure, the compressive circumferential force tends to increase in magnitude and instability may occur for large a/t ratios. Thus extreme case must be exercised by the engineer to avoid buckling failure. The ASME VIII-2 code contains design rules that take into account the instability of shallow ellipsoidal shells due to internal pressure as discussed in Section 8-6.

For spherical shells under axisymmetric loads, the differential equations can be simplified by letting $r_1 = r_2 = R$. Equations (8-11) and (8-12) become

$$N_\phi + N_\theta = p_r R \quad (8-13)$$

and

$$N_\phi = \frac{R}{\sin^2 \phi} \left[\int (p_r \cos \phi - p_\phi \sin \phi) \sin \phi \, d\phi + C \right]. \quad (8-14)$$

These two expressions form the basis for developing solutions to various loading conditions in spherical shells. For any loading condition, expressions for p_r and p_ϕ are first determined and then the above equations are solved for N_ϕ and N_θ .

Example 8-2

A concrete dome with thickness t has a dead load of γ psf. Find the expressions for N_ϕ and N_θ .

Solution

From Fig. 8-9a and Eq. (8-14),

$$p_r = -\gamma \cos \phi \quad \text{and} \quad p_\phi = \gamma \sin \phi$$

$$N_\phi = \frac{R}{\sin^2 \phi} \left[\int (-\gamma \cos^2 \phi - \gamma \sin^2 \phi) \sin \phi \, d\phi + C \right]$$

$$N_\phi = \frac{R}{\sin^2 \phi} (\gamma \cos \phi + C). \quad (1)$$

As ϕ approaches zero, the denominator in Eq. (1) approaches zero. Accordingly, we must let the bracketed term in the numerator equal zero. This yields $C = -\gamma$. Equation (1) becomes

$$N_\phi = \frac{-R\gamma(1 - \cos \phi)}{\sin^2 \phi}. \quad (2)$$

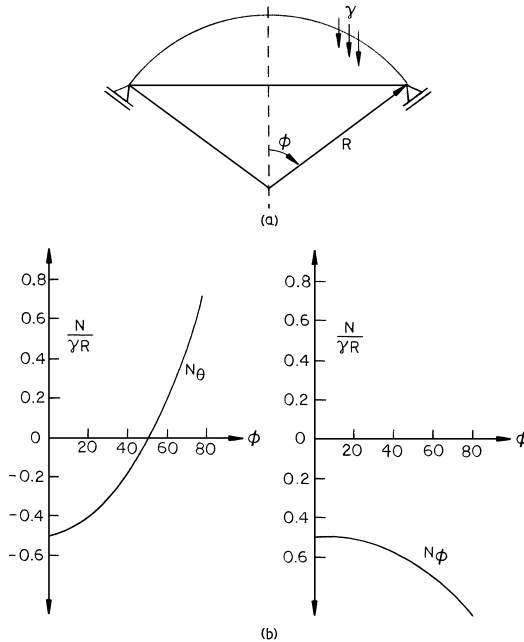


Figure 8-9. Membrane forces due to dead load of a dome.

The convergence of Eq. (2) as ϕ approaches zero can be checked by l'Hopital's rule. Thus,

$$N_\phi|_{\phi=0} = \frac{-R\gamma \sin \phi}{2 \sin \phi \cos \phi} \bigg|_{\phi=0} = \frac{-\gamma R}{2}$$

Equation (2) can be written as

$$N_\phi = \frac{-\gamma R}{1 + \cos \phi}. \quad (3)$$

From Eq. (8-13), N_θ is given by

$$N_\theta = \gamma R \left(\frac{1}{1 + \cos \phi} - \cos \phi \right). \quad (4)$$

A plot of N_ϕ and N_θ for various values of ϕ is shown in Fig. 8-9b. The plot shows that for angles ϕ greater than 52° , the hoop force, N_θ , is in tension and special attention is needed for concrete reinforcing details.

Example 8-3

Find the forces in a spherical dome due to a lantern load P_o applied at an angle $\phi = \phi_o$ as shown in Fig. 8-10a.

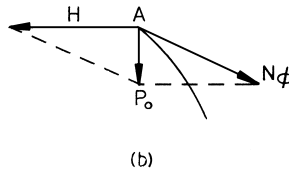
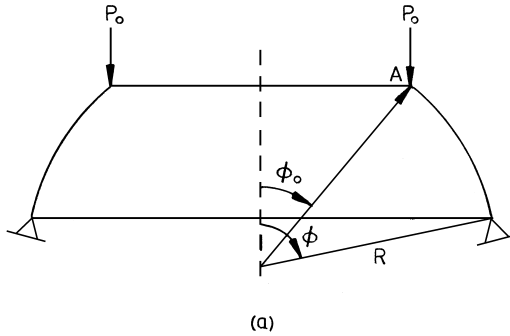


Figure 8-10. Edge load in a spherical dome.

Solution

Since $p_r = p_\phi = 0$, Eq. (8-14) becomes

$$N_\phi = \frac{RC}{\sin^2 \phi}. \quad (1)$$

From statics at $\phi = \phi_o$, we get from Fig. 8-10b

$$N_\phi = \frac{P_o}{\sin \phi_o}.$$

Substituting this expression into Eq. (1), and keeping in mind that it is a compressive membrane force, gives

$$C = \frac{-P_o}{R} \sin \phi_o$$

and Eq. (1) yields

$$N_\phi = -P_o \frac{\sin \phi_o}{\sin^2 \phi}.$$

From Eq. (8-13),

$$N_\theta = P_o \frac{\sin \phi_o}{\sin \phi}.$$

In this example there is another force that requires consideration. Referring to Fig. 8-10b, it is seen that in order for P_o and N_ϕ to be in equilibrium, another horizontal force, H , must be considered. The direction of H is inwards in order for the force system to have a net resultant force P_o downwards. This horizontal force is calculated as

$$H = \frac{-P_o \cos \phi_o}{\sin \phi_o}.$$

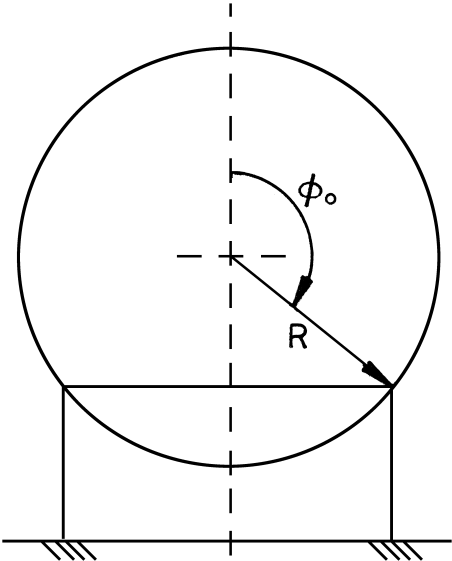
A compression ring is needed at the inner edge in order to contain force H .

Example 8-4

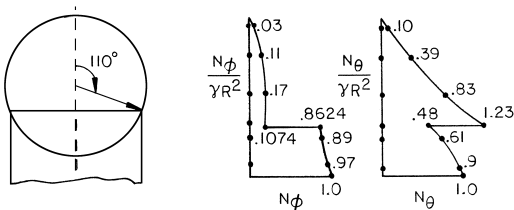
The sphere shown in Fig. 8-11a is filled with a liquid of density γ . Hence, p_r and p_ϕ can be expressed as

$$\begin{aligned} p_r &= \gamma R (1 - \cos \phi) \\ p_\phi &= 0. \end{aligned}$$

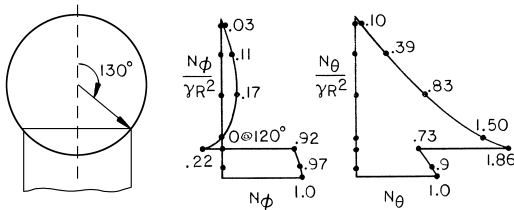
- Determine the expressions for N_ϕ and N_θ throughout the sphere.
- Plot N_ϕ and N_θ for various values of ϕ when $\phi_o = 110^\circ$.
- Plot N_ϕ and N_θ for various values of ϕ when $\phi_o = 130^\circ$.
- If $\gamma = 62.4$ pcf, $R = 30$ ft, and $\phi_o = 110^\circ$, determine the magnitude of the unbalanced force H at the cylindrical shell junction. Design the sphere, support cylinder, and the junction ring. Let the allowable stress in tension be 20 ksi and that in compression be 10 ksi.



(a)



(b)



(c)

Figure 8-11. Spherical tank.

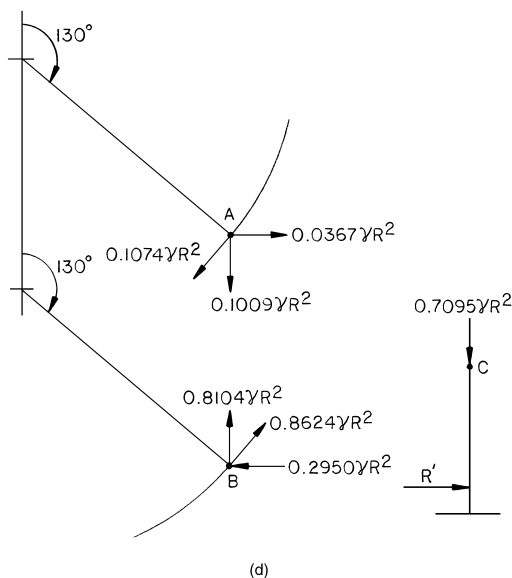


Figure 8-11. (continued)

Solution

(a) From Eq. (8-14), we obtain

$$N_\phi = \frac{\gamma R^2}{\sin^2 \phi} \left(\frac{1}{2} \sin^2 \phi + \frac{1}{3} \cos^3 \phi + C \right). \quad (1)$$

As ϕ approaches zero, the denominator approaches zero. Hence, the bracketed term in the numerator must be set to zero. This gives $C = -1/3$ and Eq. (1) becomes

$$N_\phi = \frac{\gamma R^2}{6 \sin^2 \phi} (3 \sin^2 \phi + 2 \cos^3 \phi - 2). \quad (2)$$

The corresponding N_θ from Eq. (8-13) is

$$N_\theta = \gamma R^2 \left[\frac{1}{2} - \cos \phi - \frac{1}{3 \sin^2 \phi} (\cos^3 \phi - 1) \right]. \quad (3)$$

As ϕ approaches π , we need to evaluate Eq. (1) at that point to ensure a finite solution. Again the denominator approaches zero and the bracketed term in the numerator must be set to zero. This gives $C = 1/3$ and Eq. (1) becomes

$$N_\phi = \frac{\gamma R^2}{6 \sin^2 \phi} (3 \sin^2 \phi + 2 \cos^3 \phi + 2). \quad (4)$$

The corresponding N_θ from Eq. (8-13) is

$$N_\theta = \gamma R^2 \left[\frac{1}{2} - \cos \phi - \frac{1}{3 \sin^2 \phi} (\cos^3 \phi + 1) \right]. \quad (5)$$

Equations (2) and (3) are applicable between $0 < \phi < \phi_o$, and Eqs. (4) and (5) are applicable between $\phi_o < \phi < \pi$.

(b) A plot of Eqs. (2) through (5) for $\phi_o = 110^\circ$ is shown in Fig. 8-11b. N_ϕ below circle $\phi_o = 110^\circ$ is substantially larger than that above circle 110° . This is due to the fact that most of the weight of the contents is supported by the spherical portion that is below the circle $\phi_o = 110^\circ$. Also, because N_ϕ does not increase in proportion to the increase in pressure as ϕ increases, Eq. (8-13) necessitates a rapid increase in N_θ in order to maintain the relationship between the left- and righthand sides. This is illustrated in Fig. 8-11b.

A plot of N_ϕ and N_θ for $\phi_o = 130^\circ$ is shown in Fig. 8-11c. In this case, N_ϕ is in compression just above the circle $\phi_o = 130^\circ$. This indicates that as the diameter of the supporting cylinder gets smaller, the weight of the water above circle $\phi_o = 130^\circ$ must be supported by the sphere in compression. This results in a much larger N_θ value just above $\phi_o = 130^\circ$. Buckling of the sphere becomes a consideration in this case.

(c) From Fig. 8-11b for $\phi_o = 110^\circ$, the maximum force in the sphere is $N_\theta = 1.23 \gamma R^2$. The required thickness of the sphere is

$$t = \frac{1.23(62.4)(30)^2/12}{20,000}$$

$$= 0.29 \text{ inch.}$$

A free body diagram of the spherical and cylindrical junction at $\phi_o = 110^\circ$ is shown in Fig. 8-11d. The values of N_ϕ at points *A* and *B* are obtained from Eqs. (2) and (4), respectively. The vertical and horizontal components of these forces are shown at points *A* and *B* in Fig. 8-11d. The unbalanced vertical forces result in a downward force at point *C* of magnitude $0.7095 \gamma R^2$. The total force on the cylinder is $(0.7095 \gamma R^2)(2\pi)(R)(\sin(180-110))$. This total force is equal to the total weight of the contents in the sphere given by $(4/3)(\pi R^3)\gamma$. The required thickness of the cylinder is

$$t = \frac{0.7095(62.4)(30)^2/12}{10,000}$$

$$= 0.33 \text{ inch.}$$

Summation of horizontal forces at points *A* and *B* results in a compressive force of magnitude $0.2583 \gamma R^2$. The needed area of compression ring at the cylinder to sphere junction is

$$A = \frac{Hr}{\sigma} = \frac{0.2583 \times 62.4 \times 30^2(30 \sin 70)}{10,000}$$

$$= 40.89 \text{ in.}^2$$

This area is furnished by a large ring added to the sphere or an increase in the thickness of the sphere at the junction.

Problems

8-1 Derive Eq. (8-3).

8-2 Derive Eq. (8-12).

8-3 Plot the values of N_ϕ and N_θ as a function of ϕ in an ellipsoidal shell with a ratio of 3:1 and subjected to an internal pressure p .

8-4 The nose of a submersible titanium vehicle is made of a 2:1 ellipsoidal shell. Calculate the required thickness due to an external pressure of 300 psi. Let $a = 30$ inches, $b = 15$ inches, and the allowable compressive stress = 10 ksi.

8-5 Determine the forces in the spherical shelter due to snow load.

8-6 Determine the values of N_ϕ and N_θ of the roof of the underwater habitat. For hydrostatic pressure, let

$$p_\phi = 0$$

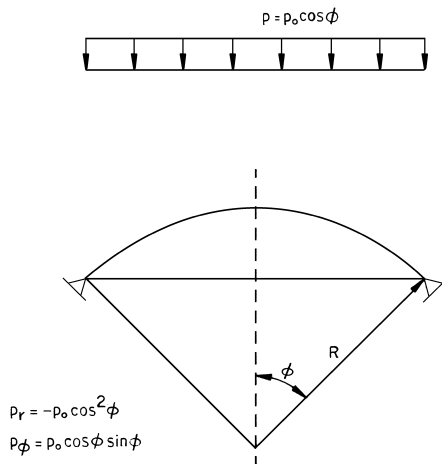
$$p_r = -\gamma[H + R(1 - \cos \phi)]$$

8-7 Determine the magnitude of the reaction R in the dome shelter shown due to dead load γ .

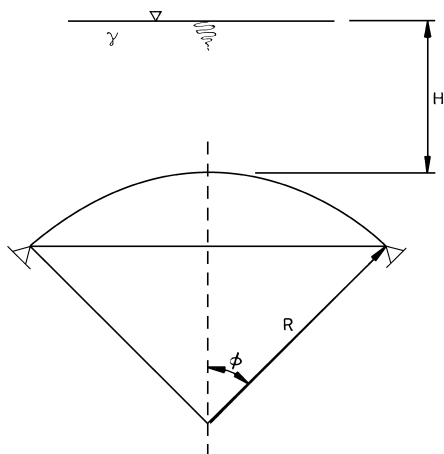
8-8 Determine the forces in the missile head due to load, γ , induced by acceleration. The equivalent pressure is expressed as

$$p_\phi = \gamma \sin \phi$$

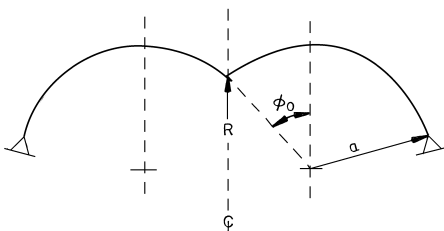
$$p_r = -\gamma \cos \phi.$$



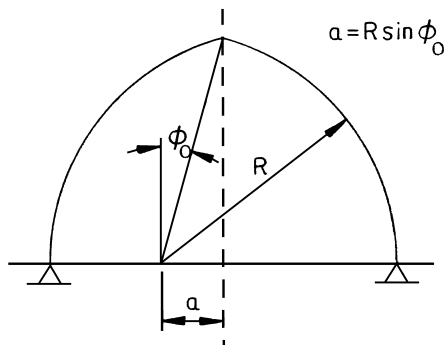
Prob. 8-5. Spherical shelter.



Prob. 8-6. Underwater habitat.



Prob. 8-7. Dome shelter.



Prob. 8-8. Missile head.

8-3 Conical Shells

Equations (8-11) and (8-12) cannot readily be used for analyzing conical shells because the angle ϕ in a conical shell is constant. Hence, the two equations have to be modified accordingly. Referring to Fig. 8-12, it can be shown that

$$\left. \begin{aligned} \phi &= \beta = \text{constant} \\ r_1 &= \infty \quad r_2 = s \tan \alpha \quad r = s \sin \alpha \\ N_\phi &= N_s \end{aligned} \right\} \quad (8-15)$$

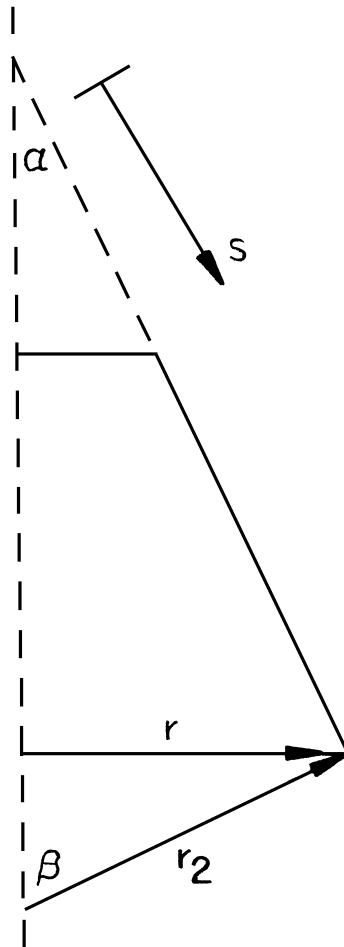


Figure 8-12. Conical shell.

Equation (8-11) can be written as

$$\frac{N_s}{r_1} + \frac{N_\theta}{s \tan \alpha} = p_r$$

or since $r_1 = \infty$,

$$\left. \begin{aligned} N_\theta &= p_r s \tan \alpha \\ &= p_r r_2 \\ N_\theta &= \frac{p_r r}{\cos \alpha} \end{aligned} \right\}. \quad (8-16)$$

Similarly from Eqs. (8-1) and (8-9),

$$\frac{d}{ds} r_1 (s \sin \alpha N_s) - r_1 N_\theta \sin \alpha = -p_s s \sin \alpha r_1.$$

Substituting Eq. (8-16) into this equation results in

$$N_s = \frac{-1}{s} \left[\int (p_s - p_r \tan \alpha) s \, ds + C \right]. \quad (8-17)$$

It is of interest to note that while N_θ is a function of N_ϕ for shells with double curvature, it is independent of N_ϕ for conical shells as shown in Eqs. (8-16) and (8-17). Also, as α approaches 0° , Eq. (8-17) becomes

$$N_\theta = p_r r_2,$$

which is the expression for the circumferential hoop force in a cylindrical shell.

The analysis of conical shells consists of solving the forces in Eqs. (8-16) and (8-17) for any given loading condition. The thickness is then determined from the maximum forces and a given allowable stress.

Example 8-5

Determine the longitudinal and circumferential forces, N_s and N_θ , of the mushroom-like concrete shelter shown in Fig. 8-13a due to a dead load, γ .

Solution

From Fig. 8-13, $p_s = \gamma \cos \alpha$ and $p_r = -\gamma \sin \alpha$

From Eq. (8-16),

$$N_\theta = -\gamma s \sin \alpha \tan \alpha.$$

From Eq. (8-17),

$$\begin{aligned} N_s &= \frac{-1}{s} \left[\int \left(\gamma \cos \alpha + \gamma \sin \alpha \frac{\sin \alpha}{\cos \alpha} \right) s \, ds + C \right] \\ &= -\frac{\gamma s}{2 \cos \alpha} - \frac{C}{s}. \end{aligned}$$

At $s = L$, $N_s = 0$.

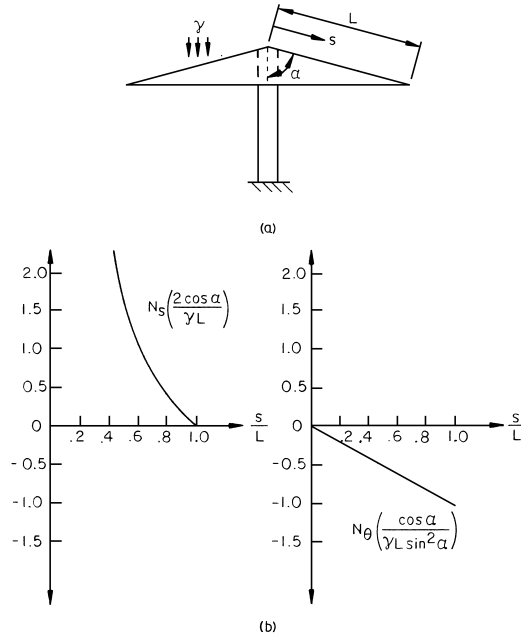


Figure 8-13. Concrete shelter.

Hence,

$$C = \frac{-\gamma L^2}{2 \cos \alpha}$$

and

$$N_s = \frac{\gamma(L^2 - s^2)}{2s \cos \alpha}.$$

A plot of N_θ and N_s is shown in Fig. 8-13b. The figure illustrates the need for longitudinal reinforcing bars along the entire length of the shelter and especially near the support. Theoretically, reinforcing bars are not needed in the hoop direction. They are needed, however, as temperature reinforcement.

Example 8-6

Determine the maximum longitudinal and circumferential forces in the conical hopper shown in Fig. 8-14 due to internal pressure p .

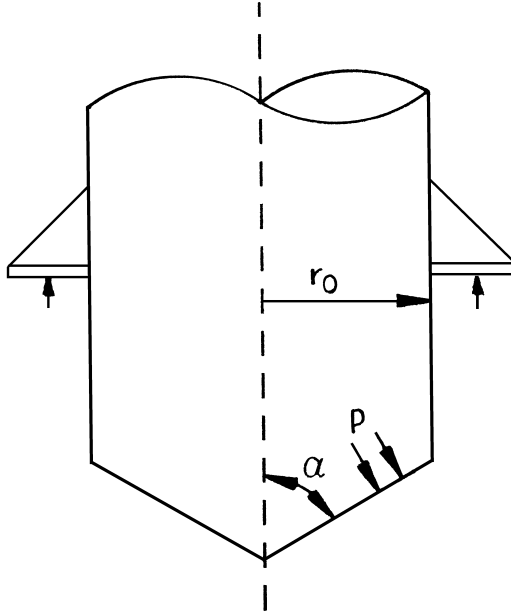


Figure 8-14. Conical Hopper.

Solution

From Eq. (8-16), the maximum N_θ occurs at the large end of the cone and is given by

$$N_\theta = p \left(\frac{r_0}{\sin \alpha} \right) \tan \alpha = \frac{pr_0}{\cos \alpha}.$$

From Eq. (8-17),

$$\begin{aligned} N_s &= \frac{-1}{s} \left(\int -p \tan \alpha \, s \, ds + C \right) \\ &= \frac{-1}{s} \left(-p \tan \alpha \frac{s^2}{2} + C \right). \end{aligned} \quad (1)$$

$$\text{At } s = L, N_s = \frac{pr_0}{2} \frac{1}{\cos \alpha}$$

Substituting this expression into Eq. (1), and using the relationships of Eq. (8-15), gives $C = 0$. Equation (1) becomes

$$\begin{aligned} N_s &= \frac{pr}{2 \cos \alpha} \\ \text{and max } N_s &= \frac{pr_0}{2 \cos \alpha}. \end{aligned}$$

It is of interest to note that the longitudinal and hoop forces are identical to those of a cylinder with equivalent radius of $r_0/\cos \alpha$.

Problems

8-9 Determine the maximum values and location of N_s and N_θ in the wine glass shown. Let $L = 3.00$ inches, $L_o = 0.25$ inches, $\gamma = 0.0289$ lb/in³, and $\alpha = 30^\circ$.

8-10 The lower portion of a reactor is subjected to a radial nozzle load as shown. Determine the required thickness of the conical section. Use an allowable stress of 10 ksi. What is the required area of the ring at the point of application of the load?

8-11 Solve Example 8-5 for a snow load, q .

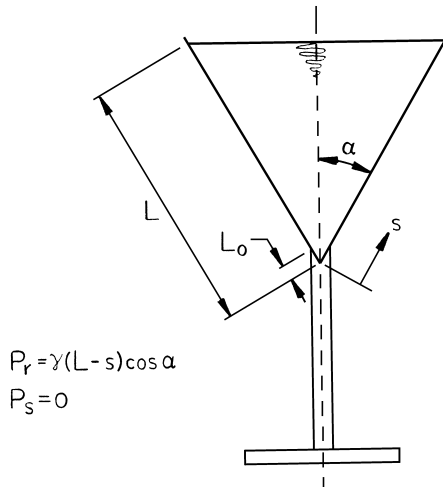
8-4 Cylindrical Shells

Equipment consisting of thin cylindrical shells subjected to pressure and axial loads are frequently encountered in refineries and chemical plants. If the radius of the shell is designated by R , Fig. 8-15a, then from Fig. 8-3 $r_1 = \infty$, $\phi = 90^\circ$, and $r = r_2 = R$. The value of the circumferential force N_θ can be obtained by equating the pressure acting on the cross section, Fig. 8-18b, to the forces in the material at the cross section. This results in

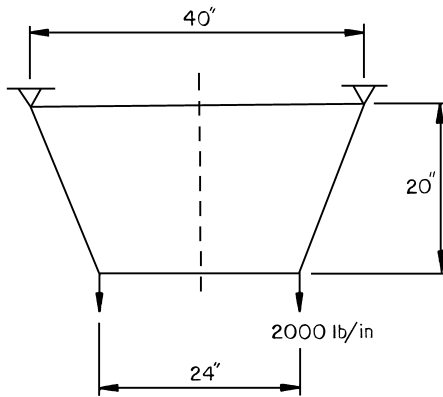
$$N_\theta = p_r R \quad (8-18)$$

Similarly, the value of the axial force N_ϕ is obtained by equating the pressure acting on the cross section, Fig. 8-18b, to the forces in the material at the cross section. This yields

$$N_\phi = p_r R/2 \quad (8-19)$$



Prob. 8-9. Wine glass.



Prob. 8-10. Lower end of a reactor.

In many pressure vessels the value of N_ϕ at the intersection of various components is determined from a free body diagram at a given cross section. Its magnitude is a function of the applied pressure at the specific location as well as other mechanical applied loads. This procedure is explained in detail in Chapter 9.

When the cross section of a thin cylinder is elliptical, Fig. 8-16, rather than circular in shape, then the relationship

$$(x/a)^2 + (y/b)^2 = 1 \quad (8-20)$$

is incorporated in the basic equations and the value of circumferential stress N_θ becomes

$$N_\theta = \frac{p_r a^2 b^2}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{3/2}} \quad (8-21)$$

where, a , b , and ϕ are as defined in Fig. 8-16. The value of N_ϕ is obtained from a free-body diagram at a given location.

Example 8-7

A cylindrical pressure vessel is constructed of steel plates and has an internal pressure of 100 psi. The steel allowable stress is 20 ksi. Find the required thickness of the cylinder when the cross section is a) circular with a diameter of 96 inches and b) elliptical with a minor diameter of 92 inches and a major diameter of 100 inches.

Solution

a) From Eq. (8-18),

$$N_\theta = (100)(96/2) = 4800 \text{ lbs/in}$$

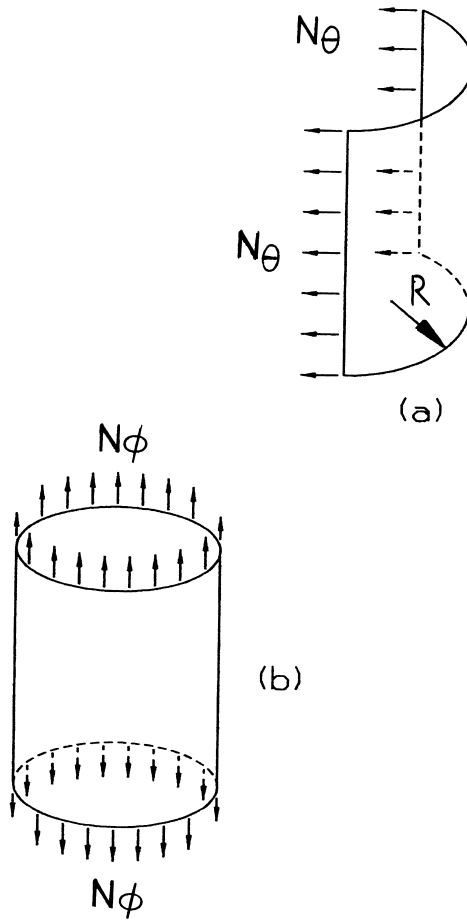


Figure 8-15. Cylindrical shell.

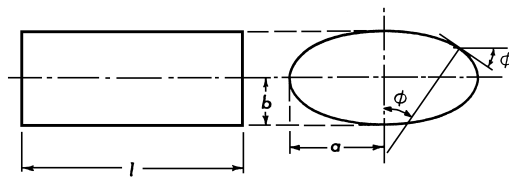


Figure 8-16. Elliptical shell.

The thickness is obtained from the relationship $t = N_\theta/\sigma$

$$t = 4800/20,000$$

or

$$t = 0.24 \text{ inch.}$$

b) From Eq. (8-21), the maximum value of N_θ occurs at $\phi = 0^\circ$.

$$N_\theta = \frac{(100)(50)^2(46)^2}{(50^2 \sin^2 0 + 46^2 \cos^2 0)^{3/2}}$$

$$N_\theta = \frac{529,000,000}{(46^2)^{3/2}}$$

$$N_\theta = 5435 \text{ lbs/inch}$$

or,

$$t = 5435/20,000$$

$$t = 0.27 \text{ inch.}$$

8-5 Wind Loads

The distribution of wind pressure on shells of revolution is assumed (Flügge 1967) perpendicular to the surface, Fig. 8-17, and is usually expressed as

$$\left. \begin{aligned} p_r &= -p \sin \phi \cos \theta \\ p_\theta &= p_\phi = 0 \end{aligned} \right\} \quad (8-22)$$

This distribution is an approximation of the actual wind pressure on a structure. More accurate approximations can be made. However, they are too complicated to solve mathematically in a closed-form equation. The result obtained from an approximate distribution is well within most acceptable engineering tolerances and is discussed in this section. The wind load, Fig. 8-17, causes a compressive pressure on the windward side of the shell and suction on the leeward side of the shell. The total sum of the pressure, i.e., windward plus leeward, is equal to the numerical value published by various standards for wind loads on structures. The load distribution is a function of both ϕ and θ and the equilibrium Eqs. (8-3), (8-7), and (8-8) are valid for obtaining membrane forces in the shell due to wind loads. Substituting Eq. (8-22) into Eq. (8-8) gives

$$N_\theta = -pr_2 \sin \phi \cos \theta - \frac{r_2}{r_1} N_\phi. \quad (8-23)$$

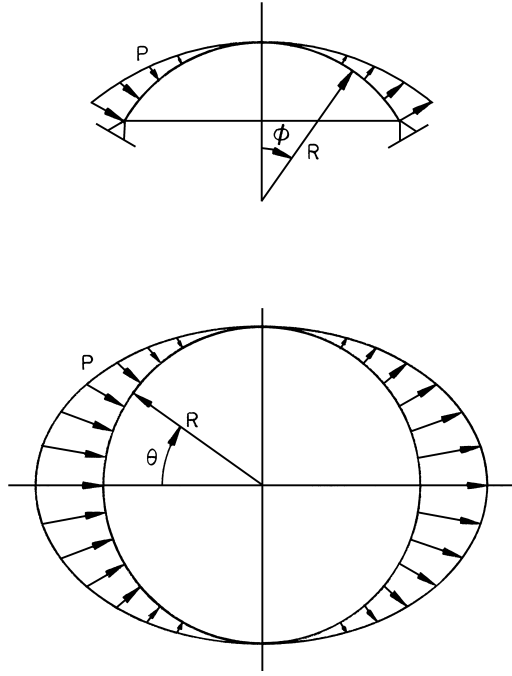


Figure 8-17. Wind pressure on a spherical shell.

Using this equation to eliminate N_θ from Eqs. (8-3) and (8-7) and using $N_{\theta\phi} = N_{\phi\theta}$ results in

$$\frac{\partial N_\phi}{\partial \phi} + \left(1 + \frac{r_1}{r_2}\right) N_\phi \cot \phi - \frac{r_1}{r} \frac{\partial N_{\phi\theta}}{\partial \theta} = -pr_1 \cos \phi \cos \theta \quad (8-24)$$

and

$$\frac{\partial N_{\phi\theta}}{\partial \phi} + 2 \frac{r_1}{r_2} N_{\phi\theta} \cot \phi + \frac{1}{\sin \phi} \frac{\partial N_\phi}{\partial \theta} = pr_1 \sin \theta. \quad (8-25)$$

Thus, membrane forces in shells of revolution due to wind loads are obtained by solving first Eqs. (8-24) and (8-25) for the two forces N_ϕ and $N_{\phi\theta}$. Then Eq. (8-23) is solved for N_θ .

For spherical shells, $r_1 = r_2 = R$. Also, the membrane forces are assumed distributed so that N_θ and N_ϕ have their maximum value at $\theta = 0^\circ$ and 180° , as these are the locations for maximum wind load, Fig. 8-17. The shell acts similar

to a beam under bending. The shearing force $N_{\phi\theta}$ is a maximum at $\theta = 90^\circ$ where N_ϕ and N_θ are zero. The three membrane forces can be expressed as

$$\left. \begin{aligned} N_\phi &= C_\phi \cos \theta \\ N_{\phi\theta} &= C_{\phi\theta} \sin \theta \\ N_\theta &= C_\theta \cos \theta \end{aligned} \right\} \quad (8-26)$$

where C_ϕ , $C_{\phi\theta}$, and C_θ are functions of ϕ only. Substituting Eq. (8-26) into Eq. (8-24) results in

$$\frac{dC_\phi}{d\phi} + 2C_\phi \cot \phi - \frac{1}{\sin \phi} C_{\phi\theta} = -pR \cos \phi. \quad (8-27)$$

Similarly, substituting Eq. (8-26) into Eq. (8-25) gives

$$\frac{dC_{\phi\theta}}{d\phi} + 2C_{\phi\theta} \cot \phi - \frac{1}{\sin \phi} C_\phi = pR. \quad (8-28)$$

The solution of Eqs. (8-27) and (8-28) can best be obtained by defining functions U_ϕ and V_ϕ as

$$\left. \begin{aligned} U_\phi &= C_\phi + C_{\phi\theta} \\ V_\phi &= C_\phi - C_{\phi\theta} \end{aligned} \right\} \quad (8-29)$$

Adding Eqs. (8-27) and (8-28) gives

$$\frac{dU_\phi}{d\phi} + \left(2 \cot \phi - \frac{1}{\sin \phi} \right) U_\phi = pR(1 - \cos \phi). \quad (8-30)$$

This is a partial differential equation of the form

$$\frac{dy}{dx} + A(x)y = B(x),$$

the solution of which is given by

$$y = \frac{1}{e^{\int A(x) dx}} \int B(x) e^{\int A(x) dx} dx.$$

The quantity $e^{\int A(x) dx} = e^{\int (2 \cot \phi - (1/\sin \phi)) dx}$

$$= e^{2 \ln \sin \phi - \ln \tan(\phi/2)}$$

$$= \sin^2 \phi \cot(\phi/2).$$

The quantity $\int B(x) e^{\int A(x) dx} = \int_0^\phi pR(1 - \cos \phi) \sin^2 \phi \cot(\phi/2)$

$$= \int_0^\phi pR \sin^3 \phi$$

$$= \frac{-pR}{3} [\cos \phi (\sin^2 \phi + 2) - 2].$$

Thus, U_ϕ becomes

$$U_\phi = \frac{-pR}{3 \sin^2 \phi \cot(\phi/2)} [\cos \phi (\sin^2 \phi + 2) - 2].$$

Similarly, V_ϕ in Eq. (8-29) is obtained by subtracting Eqs. (8-27) and (8-28) and following the above procedure. This gives

$$V_\phi = \frac{pR}{3 \sin^2 \phi \tan(\phi/2)} [\cos \phi (\sin^2 \phi + 2) - 2]$$

and

$$\begin{aligned} C_\phi &= \frac{U_\phi + V_\phi}{2} \\ &= \frac{pR [\cos \phi (\sin^2 \phi + 2) - 2] \cos \phi}{3 \sin^3 \phi}. \end{aligned}$$

Similarly

$$\begin{aligned} C_{\phi\theta} &= \frac{U_\phi - V_\phi}{2} \\ &= \frac{pR [\cos \phi (\sin^2 \phi + 2) - 2]}{3 \sin^3 \phi}. \end{aligned}$$

With C_ϕ and $C_{\phi\theta}$ known, Eq. (8-26) is solved for N_ϕ and $N_{\phi\theta}$. The value of N_θ is obtained from Eq. (8-23).

For conical shells, Fig. 8-12 and Eq. (8-15), define the geometry. Replacing $N_{\phi\theta}$ with $N_{s\theta}$ and defining $ds = r d\phi$, we get from Eq. (8-25),

$$\frac{dN_{s\theta}}{ds} + \frac{2}{s} N_{s\theta} = p \sin \theta$$

which upon integrating gives

$$N_{s\theta} = \frac{p \sin \theta}{s^2} \left(\frac{s^3}{3} + C \right) \quad (8-31)$$

where C is a constant obtained from the boundary conditions. Similarly, Eq. (8-24) reduces to

$$N_s = p \left(\frac{s}{6 \sin \alpha} - \frac{C}{s^2 \sin \alpha} - \frac{s \sin \alpha}{2} \right) \cos \theta \quad (8-32)$$

and Eq. (8-23) for the hoop force becomes

$$N_\theta = -ps \sin \alpha \cos \theta. \quad (8-33)$$

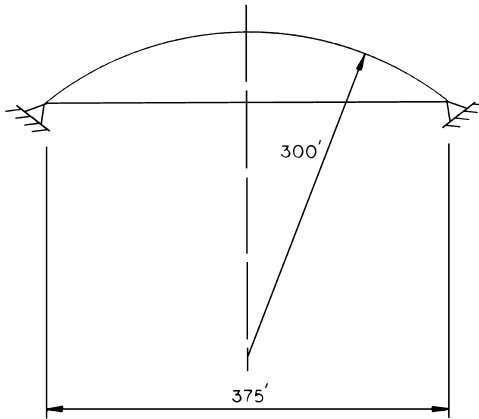
Problems

8-12 Derive Eqs. (8-24) and (8-25).

8-13 Derive Eqs. (8-31) and (8-32).

8-14 The auditorium dome is subjected to a wind pressure, p , of 12 psf. Determine the maximum forces and their location.

8-15 The picnic shed is subjected to a wind pressure, p , of 50 kgf/m². Determine the maximum forces and their location.

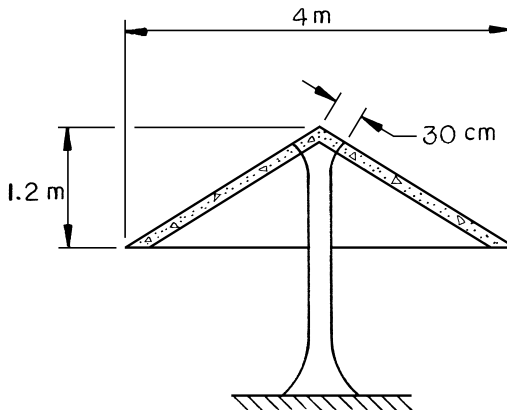


Prob. 8-14. Auditorium dome.

8-6 Design of Shells of Revolution

The maximum forces for various shell geometries subjected to commonly encountered loading conditions are listed in numerous references. One such reference is by NASA (Baker et al. 1968), where extensive tables and design charts are listed. Flugge (1967) contains a thorough coverage of a wide range of applications to the membrane theory, as does Roark and Young (1975).

Extra care should be taken in the design of shallow ellipsoidal heads subjected to internal pressure. Example 8-1 illustrated the possibility of instability in the circumferential direction of an ellipsoidal head due to internal pressure. Many codes have provisions and design aids for avoiding such instability, which tends



Prob. 8-15. Picnic shed.

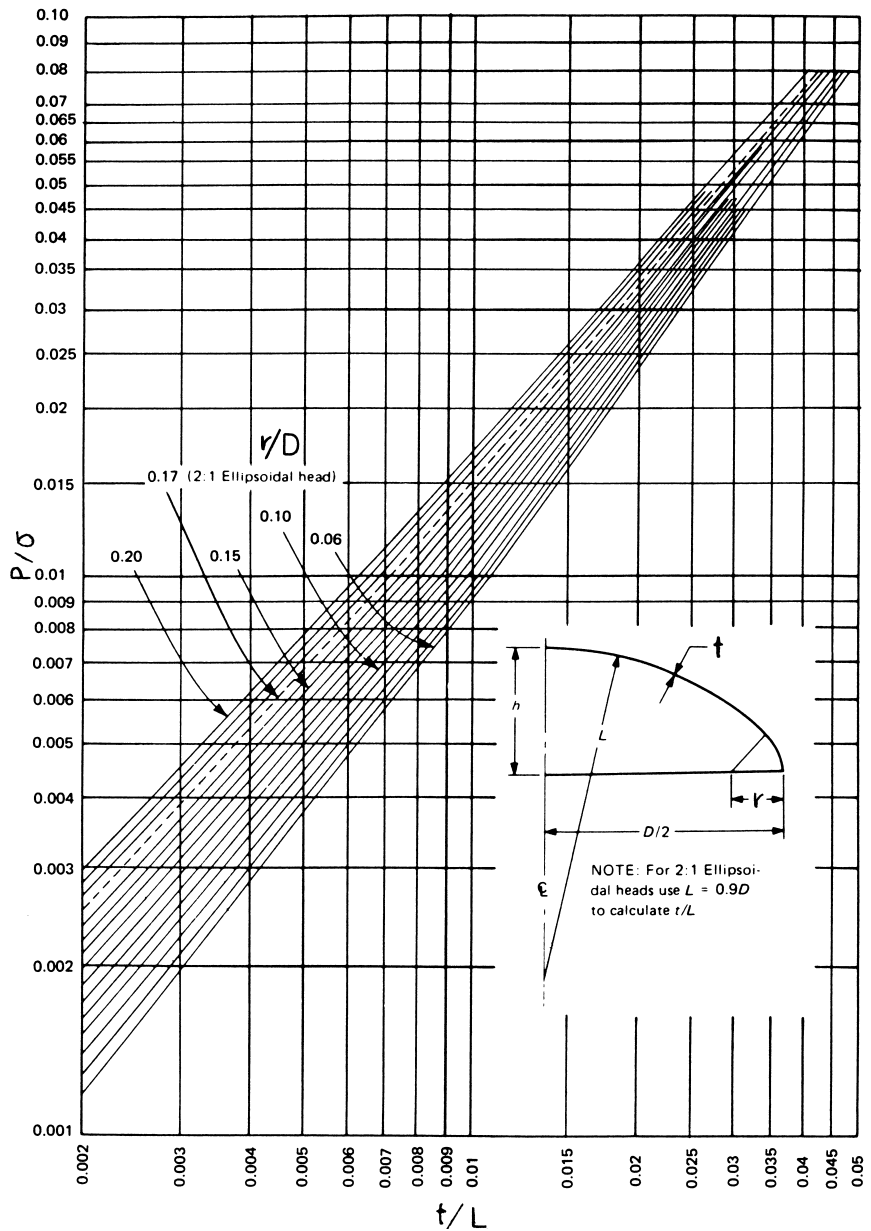


Figure 8-18. Design chart for ellipsoidal heads. (Courtesy of ASME.)

to occur in heads subjected to low pressures and having large diameter-to-thickness ratios of over about 230. The ASME VIII-2 code provides a chart, Fig. 8-18, for designing ellipsoidal heads with various a/b ratios. The design is based on approximating the geometry of a head with a spherical radius, L , and a knuckle radius, r , as defined in Fig. 8-18. The required thickness of a specific head is determined from Fig. 8-18 by knowing the values of L , r , base diameter of the head, applied pressure, and allowable membrane stress.

Figure 8-18 is plotted from the following equation.

$$t = Le^A$$

where

$$\begin{aligned} A &= A_1 + A_2 + A_3 \\ A_1 &= -1.26176643 - 4.5524592 (r/D) + 28.933179 (r/D)^2 \\ A_2 &= [0.66298796 - 2.2470836 (r/D) + 15.682985 (r/D)^2][\ln(P/S)] \\ A_3 &= [0.26878909 \times 10^{-4} - 0.42262179 (r/D) + 1.8878333 (r/D)^2][\ln(P/S)]^2 \\ D &= \text{base diameter, in.} \\ L &= \text{crown radius, in.} \\ P &= \text{design pressure, psi} \\ r &= \text{knuckle radius, in.} \\ S &= \text{allowable stress, in.} \\ t &= \text{thickness} \end{aligned}$$

Example 8-8

Some of the commonly used boiler heads in the United States are ellipsoidal in shape and have an a/b ratio of about 2.95. This corresponds to an approximate value, Fig. 8-18, of $L = D$ and $r = 0.06D$. Determine the required thickness of this head if $D = 14$ ft, $p = 75$ psi, and $\sigma = 20,000$ psi. Calculate the required thickness from Eq. (4) of Example 8-1 and also from Fig. 8-18.

Solution

From Eq. (4) of Example 8-1, the maximum force is at $\phi = 0^\circ$. Thus,

$$\begin{aligned} t &= \frac{pa^2}{2b\sigma} \\ &= \frac{75 \times (168/2)^2}{2 \times (84/2.95) \times 20,000} \\ &= 0.46 \text{ inch.} \end{aligned}$$

From Fig. 8-18 with $p/\sigma = 0.0038$, and $r/D = 0.06$ we get $t/L = 0.005$,
or $t = 0.005 \times 168 = 0.84$ inch (controls).

Notice that the thickness obtained from Fig. 8-18 is almost double that determined from theoretical membrane equations that do not take into consideration any instability due to internal pressure.

Various Applications of the Membrane Theory

9-1 Analysis of Multi-Component Structures

The quantity $N_\phi r_2 \sin^2 \phi$ in Eq. (8-12), when multiplied by 2π , represents the total applied force acting on a structure at a given parallel circle of angle ϕ . Hence, for complicated geometries, the value of N_ϕ in Eq. (8-12) at any given location can be obtained by taking a free-body diagram of the structure. The value of N_ϕ at the same location can then be determined from Eq. (8-11). This method is widely used (Jawad and Farr 1989) in designing pressure vessels, flat-bottom tanks, elevated water towers, Fig. 9-1, and other similar structures. Example 9-1 illustrates the application of this method to the design of a water tower. The American Petroleum Institute (API 620 1991) Standard has various equations and procedures for designing components by the free-body method. This method is also useful in obtaining an approximate design at the junction of two shells of different geometries. A more accurate analysis utilizing bending moments may then be performed to establish the discontinuity stresses of the selected members at a junction if a more exact analysis is needed.

Example 9-1

The tank shown in Fig. 9-2 is filled with a liquid up to point a . The specific gravity is 1.0. Above point a the tank is subjected to a gas pressure of 0.5 psi. Determine the forces and thicknesses of the various components of the steel tank disregarding the dead weight of the tank. Use an allowable tensile stress of 12,000 psi and an allowable compressive stress of 8000 psi.

Solution

Tank Roof

The maximum force in the roof is obtained from Fig. 9-3a. Below section $a-a$, a 0.5 psi pressure is needed to balance the pressure above section $a-a$. Force N_ϕ in

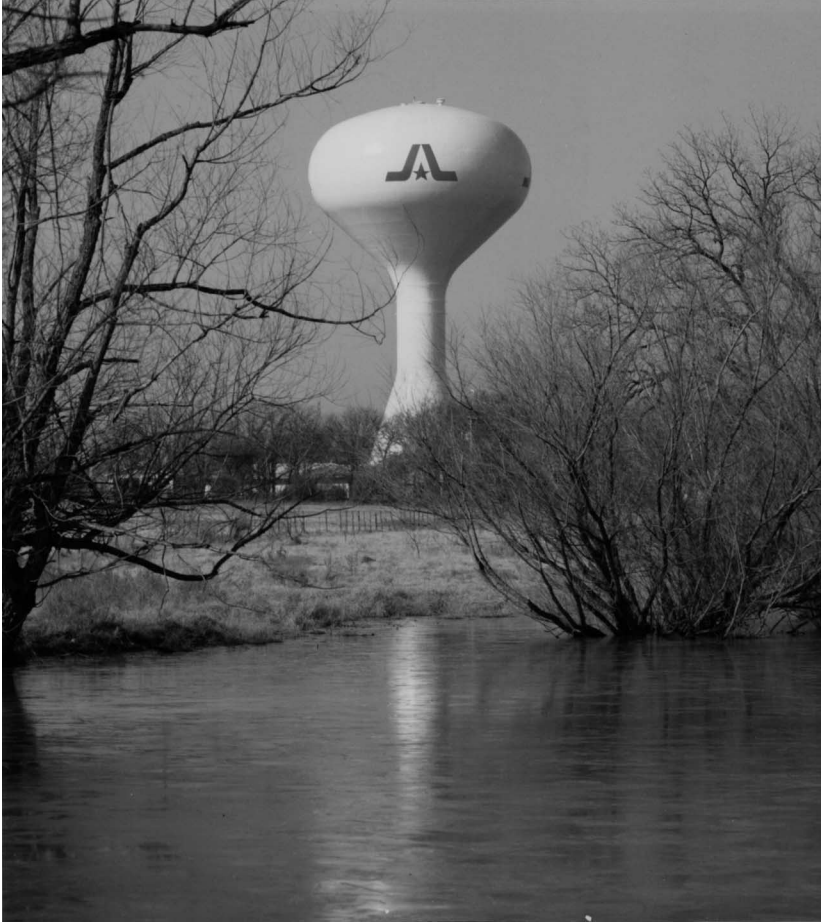


Figure 9-1. Elevated water tank. (Courtesy of CB & I)

the roof has a vertical component V around the perimeter of the roof. Summation of forces in the vertical direction gives

$$2\pi RV - \pi R^2 P = 0$$

$$V = 60 \text{ lbs/inch.}$$

Hence,

$$\begin{aligned} N_{\phi} &= \frac{V}{\sin \phi} = 60/0.42 \\ &= 144 \text{ lbs/inch.} \end{aligned}$$

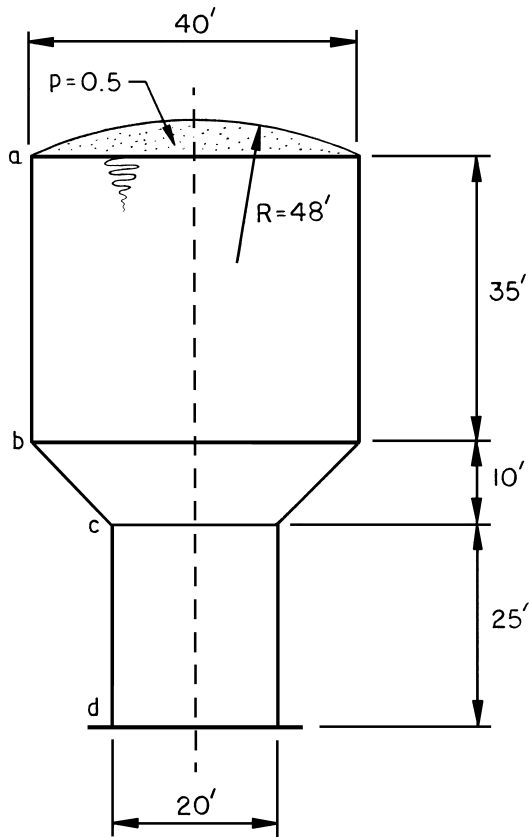


Figure 9-2. Storage tank.

From Eq. (8-11) with $r_1 = r_2 = R$

$$N_{\theta} = 144 \text{ lbs/inch.}$$

Since N_{θ} and N_{ϕ} are the same, use either one to calculate the thickness. The required thickness is

$$\begin{aligned} t &= N_{\phi} / \sigma = 144 / 12,000 \\ &= 0.012 \text{ inch.} \end{aligned}$$

Because this thickness is impractical to handle during fabrication of a tank with such a diameter, use $t = 1/4$ inch.

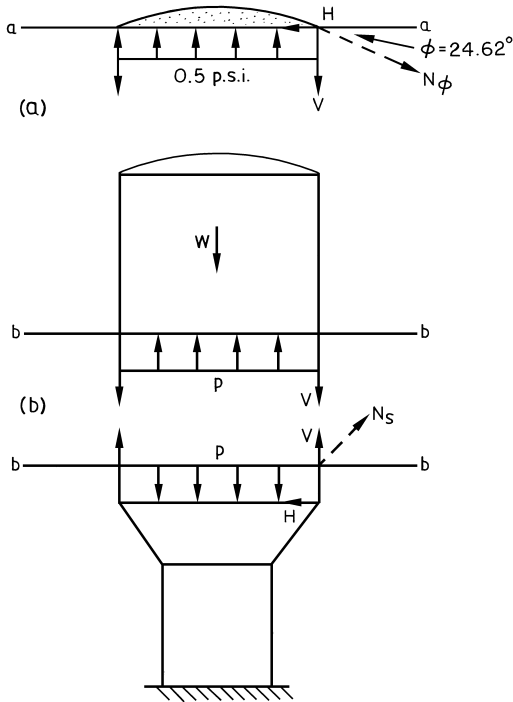


Figure 9-3. Free-body diagram of storage tank.

40-Ft Shell

The maximum force in the shell is at section $b-b$ as shown in Fig. 9-3b. The total weight of liquid at section $b-b$ is

$$\begin{aligned} W &= 62.4(\pi)(20)^2(35) \\ &= 2,744,500 \text{ lbs.} \end{aligned}$$

Total pressure at $b-b$ is

$$\begin{aligned} P &= 0.5 + (62.4/144)(35) \\ P &= 15.67 \text{ psi.} \end{aligned}$$

The total sum of the vertical forces at $b-b$ is equal to zero. Hence,

$$2,744,500 - (15.67)(\pi)(240)^2 + V(\pi)(480) = 0$$

or

$$V = 60 \text{ lbs/inch}$$

and

$$N_\phi = 60 \text{ lbs/inch.}$$

In a cylindrical shell, $r_1 = \infty$ and $r_2 = R$. Hence, Eq. (8-11) becomes

$$N_\theta = pR = (15.67)(240) = 3761 \text{ lb/inch.}$$

The required thickness $t = N_\theta/\sigma = 3761/12,000$.

$$t = .031 \text{ inch.}$$

Use $t = 3/8$ inch in order to match the conical transition section discussed later.

The unbalanced force at the roof-to-shell junction is

$$H = 131 \text{ lb/inch(inwards).}$$

The area required to contain the unbalanced force, H , is given by

$$\begin{aligned} A &= (H) (\text{shell radius})/\text{allowable compressive stress} \\ &= 131 \times 240/8000 = 3.93 \text{ inch}^2. \end{aligned}$$

Use 1 inch thick $\times$ 4 inch wide ring as shown in Fig. 9-4a.

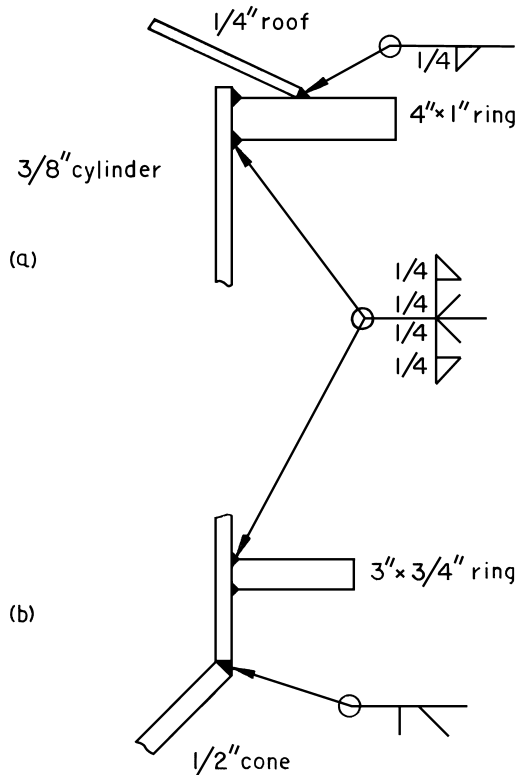


Figure 9-4. Roof-to-large cylinder, and large cylinder-to-cone junctions.

Conical Transition

At section $b-b$, force V in the 40-ft shell must equal force V in the cone in order to maintain equilibrium as shown in Fig. 9-3b. Thus,

$$V = 60 \text{ lb/inch.}$$

and

$$N_s = 60/0.707 = 85 \text{ lb/inch.}$$

In a conical shell $r_1 = \infty$ and $r_2 = R/\cos \alpha$. Hence from Eq. (8-16)

$$N_\theta = pR/\cos \alpha = 240(15.67)/0.707 = 5319 \text{ lb/inch.}$$

The required thickness at the large end of the cone is

$$t = 5319/12,000 = 0.44 \text{ inch.}$$

The horizontal force at point b is $N_s \cos 45$.

$$H = 60 \text{ lbs/inch (inwards)}$$

The required area is

$$A = 60 \times 240/8000 = 1.8 \text{ in}^2.$$

Use 3/4 inch thick by 3 inch wide ring as shown in Fig. 9-4b.

The forces at the small end of the cone are shown in Fig. 9-5a. The weight of the liquid in the conical section at point c is

$$\begin{aligned} W &= \pi \gamma H (r_1^2 + r_1 r_2 + r_2^2)/3 \\ &= \pi \times 62.4 \times 10(10^2 + 10 \times 20 + 20^2)/3 \\ &= 457,400/\text{lbs.} \end{aligned}$$

Total liquid weight is

$$\begin{aligned} W &= 2,744,500 + 457,400 \\ &= 3,201,900 \text{ lbs.} \end{aligned}$$

Pressure at section $c-c$ is

$$\begin{aligned} P &= 0.5 + (62.4/144)(45) \\ &= 20.0 \text{ psi.} \end{aligned}$$

Summing forces at section $c-c$ gives

$$\begin{aligned} 20.0 \times \pi \times 120^2 - 3,201,900 - (V \times \pi \times 240) &= 0 \\ V &= -3047 \text{ lb/inch.} \end{aligned}$$

The negative sign indicates that the vertical component of N_s is opposite to that assumed in Fig. 9-5a and is in compression rather than tension. This is caused by

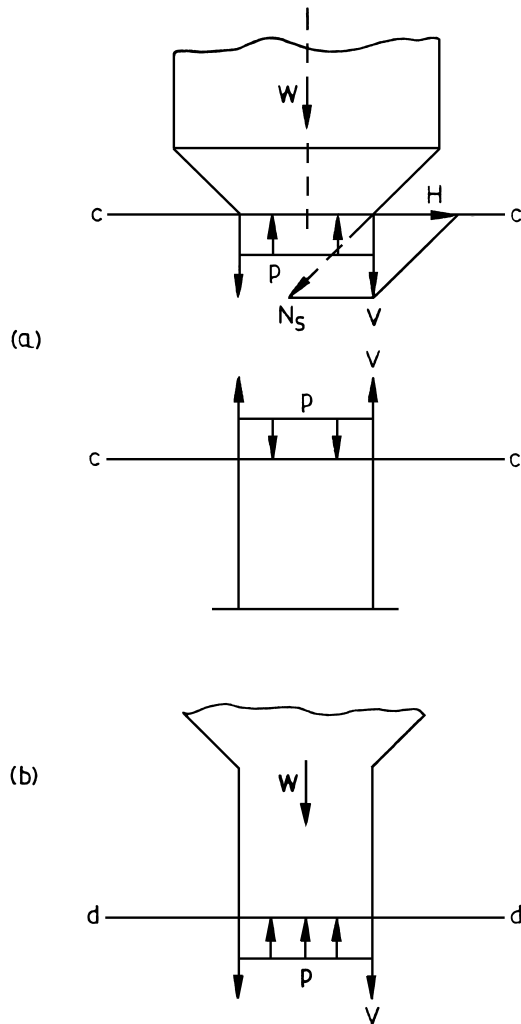


Figure 9-5. Cone-cylinder junction.

the column of liquid above the cone whose weight is greater than the net pressure force at section *c-c*.

$$\begin{aligned}
 N_s &= -3047/0.707 \\
 &= -4309 \text{ lb/inch (compressive)} \\
 N_\theta &= pR/\cos \alpha = 20.0 \times 120/0.707 \\
 &= 3395 \text{ lb/inch.}
 \end{aligned}$$

N_θ at the small end is smaller than N_θ at the large end. Hence, the thickness at the small end need not be calculated for N_θ . Since N_s at the small end is in compression, the thickness due to this force needs to be calculated because the allowable stress in compression is smaller than that in tension. Hence,

$$t = 4309/8000 = 0.54 \text{ inch.}$$

Use $t = 5/8$ inch for the cone.

The horizontal force at section $c-c$, Fig. 9-5, is given by

$$H = 3047 \text{ lb/inch inwards.}$$

The required area of the ring is

$$A = 3047 \times 120/8000 = 45.71 \text{ in.}^2$$

This large required area is normally distributed around the junction as shown in Fig. 9-6.

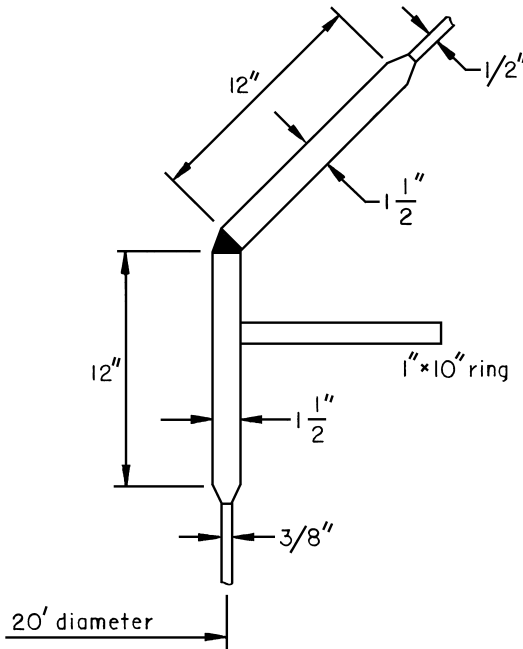


Figure 9-6. Cone-to-small cylinder junction.

20-Ft Shell

At section *c-c*, Fig. 9-5, the value of V in the 20-ft shell is the same as V in the cone due to continuity. Thus,

$$N_{\phi} = V = -3047 \text{ lb/inch}$$

$$N_{\theta} = pR = 20.0 \times 120 = 2400 \text{ lb/inch.}$$

At section *d-d*, Fig. 9-5, the liquid weight is given by

$$\begin{aligned} W &= 3,201,900 + (62.4)(\pi)(10)^2(25) \\ &= 3,692,000 \text{ lb} \end{aligned}$$

and the pressure is calculated as

$$\begin{aligned} p &= 0.5 + (62.4/144)(70) \\ &= 30.83 \text{ psi.} \end{aligned}$$

From Fig. 9-5b, the summation of forces about *d-d* gives

$$3,692,000 - 30.83 \times \pi \times 120^2 + V \times \pi \times 240 = 0$$

or

$$N_{\phi} = V = -3047 \text{ lb/inch,}$$

which is the same as that at point *c*.

$$\begin{aligned} N_{\theta} &= pR = 30.83 \times 120 \\ &= 3700 \text{ lb/inch.} \end{aligned}$$

The required thickness of the shell is governed by N_{ϕ} at section *d-d*.

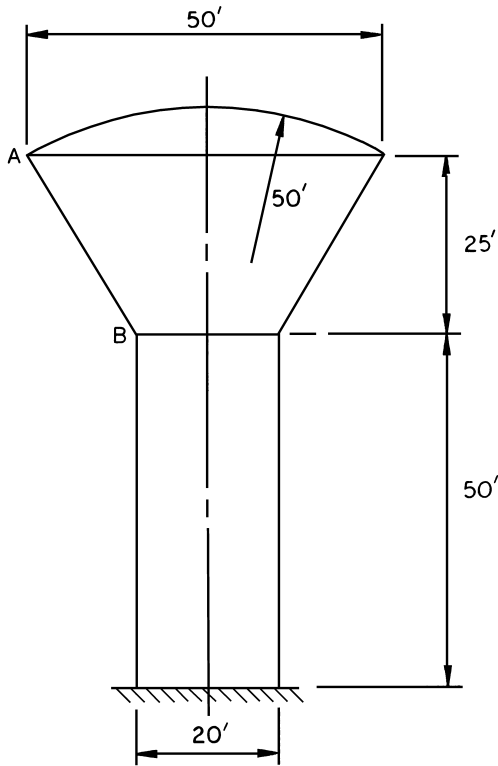
$$t = 3047/8000 = 0.38 \text{ inch.}$$

Use $t = 3/8$ inch for bottom cylindrical shell.

Problems

9-1 Determine the thickness of all components including stiffener rings at points A and B of the steel tower. The tower is full of water to point A. The roof is subjected to a snow load of 25 psf. The allowable stress in tension is 15 ksi and that in compression is 10 ksi.

9-2 Determine the thickness of all components including stiffener rings of the elevated water tower. The tower is full of liquid between points A and D. The allowable stress in tension is 15 ksi and that in compression is 10 ksi.

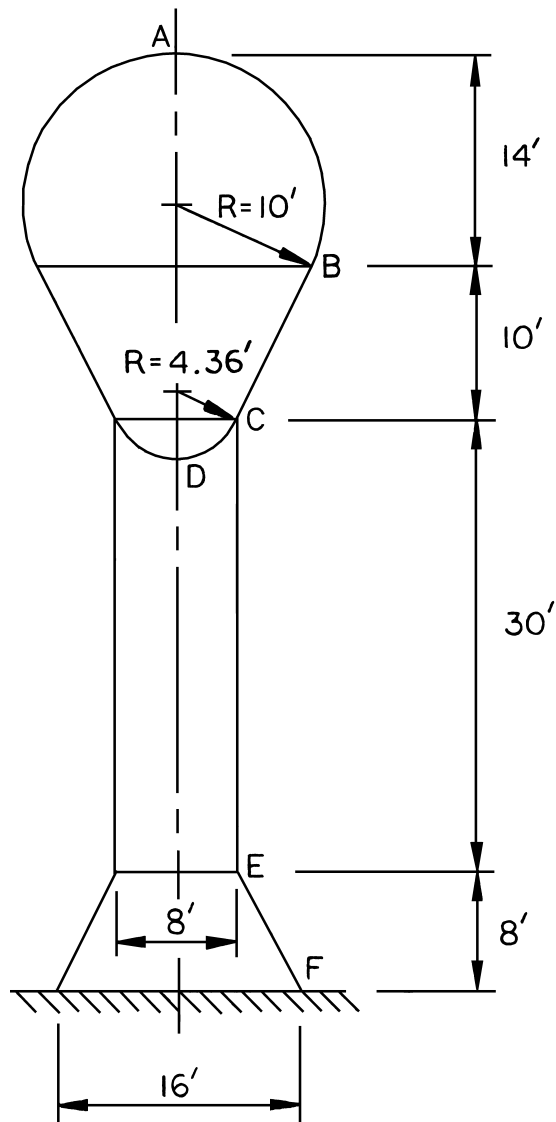


Prob. 9-1. Water tower.

9-2 Pressure-Area Method of Analysis

The membrane theory is very convenient in determining thicknesses of major components such as cylindrical, conical, hemispherical, and ellipsoidal shells. The theory, however, is inadequate for analyzing complicated geometries such as nozzle attachments, transition sections, and other details similar to those shown in Fig. 9-7. An approximate analysis of these components can be obtained by using the pressure-area method. A more accurate analysis can then be performed based on the bending theories of Chapters 10 and 11 or the finite element theory.

The pressure-area analysis is based on the concept (Zick 1963) that the pressure contained in a given area within a shell must be resisted by the metal close to that area. Referring to Fig. 9-8a, the total force in the shaded area of the cylinder is $(r)(P)(L)$ while the force supported by the available metal is $(L)(t)(\sigma)$. Equating these two expressions results in $t = Pr/\sigma$ which is the

**Prob. 9-2.** Elevated water tank.

equation for the required thickness of a cylindrical shell. Similarly for spherical shells, Fig. 9-8b gives

$$(R\phi)(R)(P)(1/2) = (R\phi)(t)(\sigma)$$

$$t = PR/2\sigma.$$

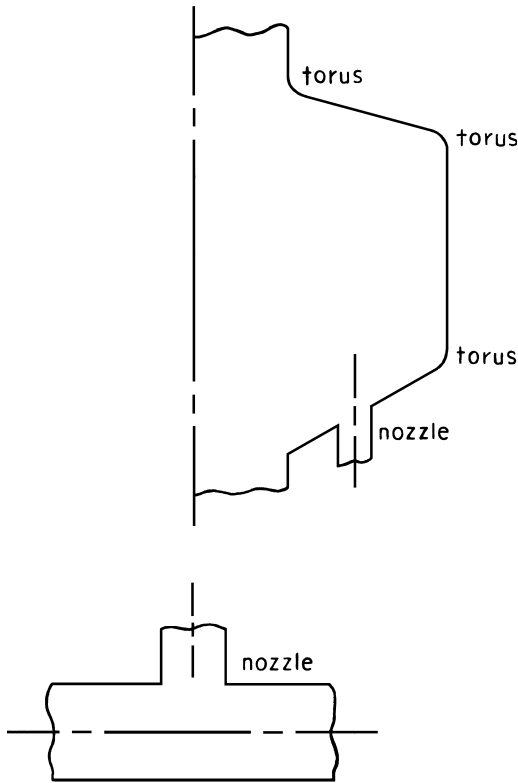


Figure 9-7. Various components.

Referring to Fig. 9-9a, it is seen that pressure area A is contained by the cylinder wall and pressure area B is contained by the nozzle wall. However, pressure area C is not contained by any material. Thus we must add material, M , at the junction. The area of material M is given by

$$(P)(R)(r) = (\sigma)(M)$$

$$M = P(R)(r)/\sigma.$$

For a spherical shell, the required area, M , from Fig. 9-9b is

$$(P)(R)(r)(1/2) = (\sigma)(M)$$

$$M = (1/2)(P)(R)(r)/\sigma.$$

The required area is added either to the shell, nozzle, or as a reinforcing pad as shown in Fig. 9-10.

The pressure-area method can also be applied for junctions (Farr and Jawad 2001) between components as shown in Fig. 9-11. Referring to Fig. 9-11a, the spherical

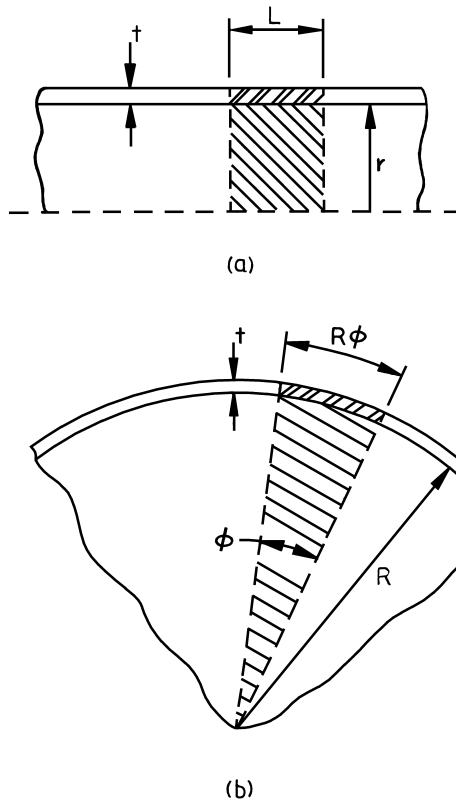


Figure 9-8. Pressure-area interaction.

shell must contain the pressure within area ABC . The cylindrical shell contains the pressure within area $AOCD$. At point A where the spherical and cylindrical shells intersect, the pressure area to be contained at point A is given by AOC . However, because area AOC is used both in the ABC area for the sphere and $AOCD$ for the cylinder, and because it can be used only once, this area must be subtracted from the total calculated pressure in order to maintain equilibrium. In other words, this area causes compressive stress at point A . The area required is given by

$$A = (r)(\sqrt{R^2 - r^2})(1/2)(P)/\sigma$$

where σ is allowed compressive stress.

In Fig. 9-11b, pressure area A is contained by the cylindrical shell and area C by the spherical shell. Area B is contained by the transition shell. The transition shell is in tension because area B is used neither in the area A nor area B calculations.

In Fig. 9-11c, pressure area A is contained by the cone and area B by the cylinder. The transition shell between the cone and the cylinder contains pressure area C

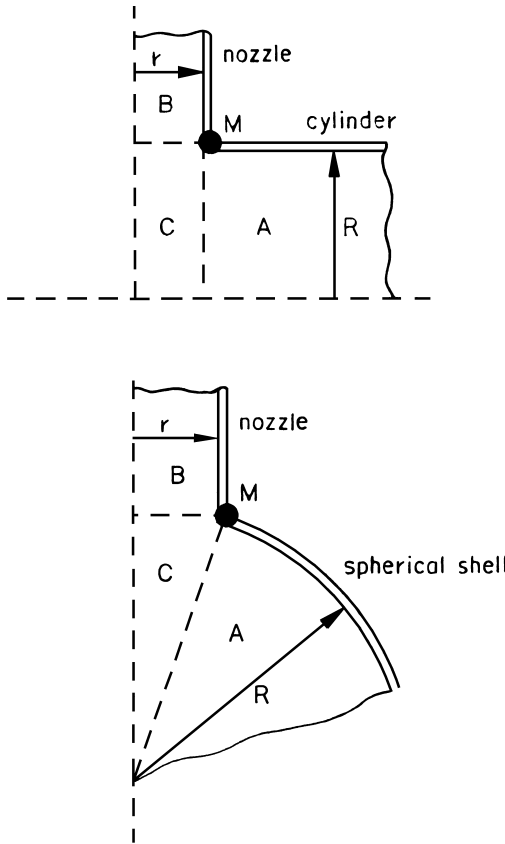


Figure 9-9. Nozzle junctions.

which is in tension and area D which is in compression. Summation of areas C and D will determine the state of stress in the transition shell.

The pressure-area method is also commonly used to design fittings and other piping components. Figure 9-12 shows one design method for some components.

Example 9-2

Find the required thickness of the cylindrical, spherical, and transition shells shown in Fig. 9-13. Let $p = 150$ psi and $\sigma = 15,000$ psi.

Solution

For the cylindrical shell,

$$t = pr/\sigma = 150 \times 36/15,000 = 0.36 \text{ inch.}$$

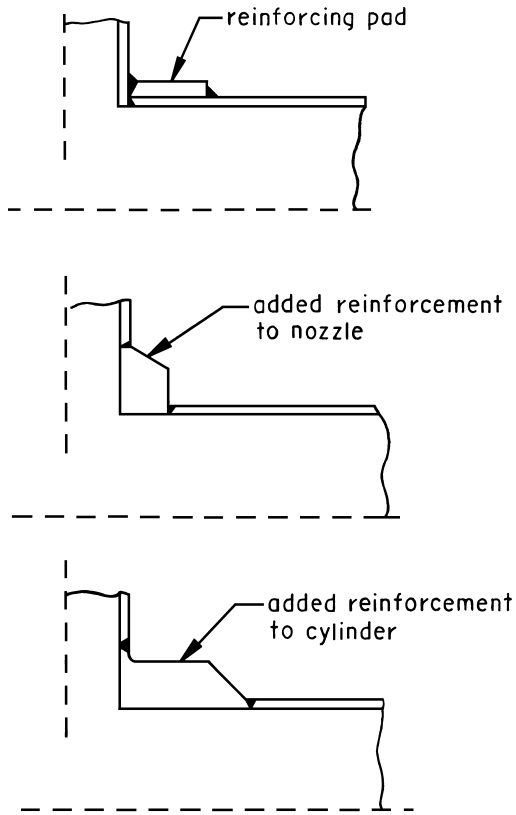


Figure 9-10. Nozzle reinforcement.

For the spherical shell,

$$t = pR/2\sigma = 150 \times 76/2 \times 15,000 = 0.38 \text{ inch.}$$

For the transition shell, pressure area B = area of triangle abc – area of segment ade

$$= 150[(76 \cos 41.14 \times 50/2 - 14^2(48.86 \times \pi/180)/2]$$

$$= 202,110 \text{ lbs-in.}^2$$

$$t \times \sigma \times 14(48.86\pi/180) = 202,110$$

or

$$t = 1.13 \text{ inch.}$$

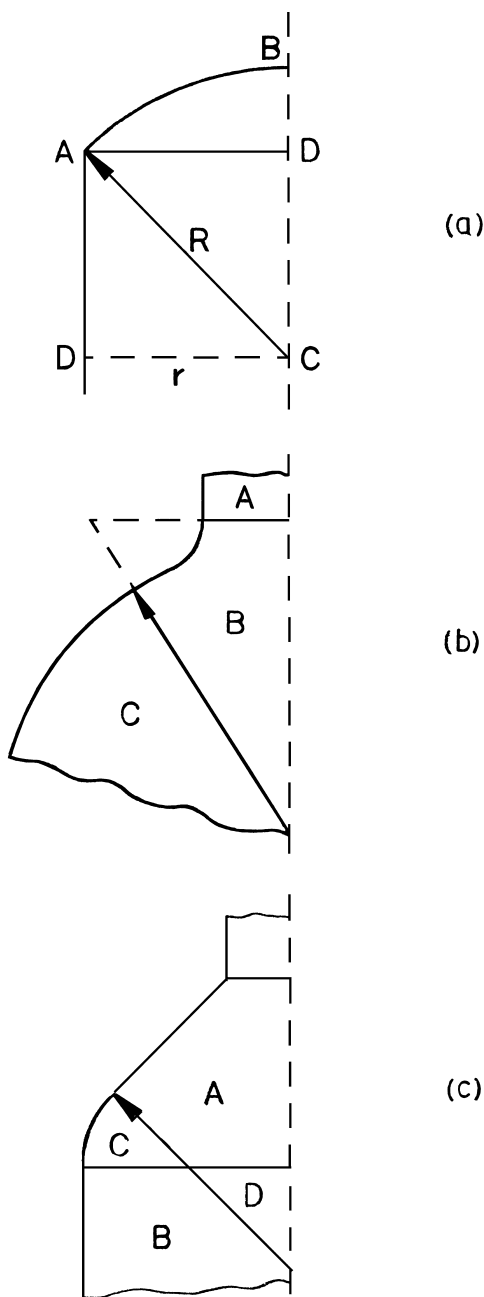


Figure 9-11. Various shell junctions.

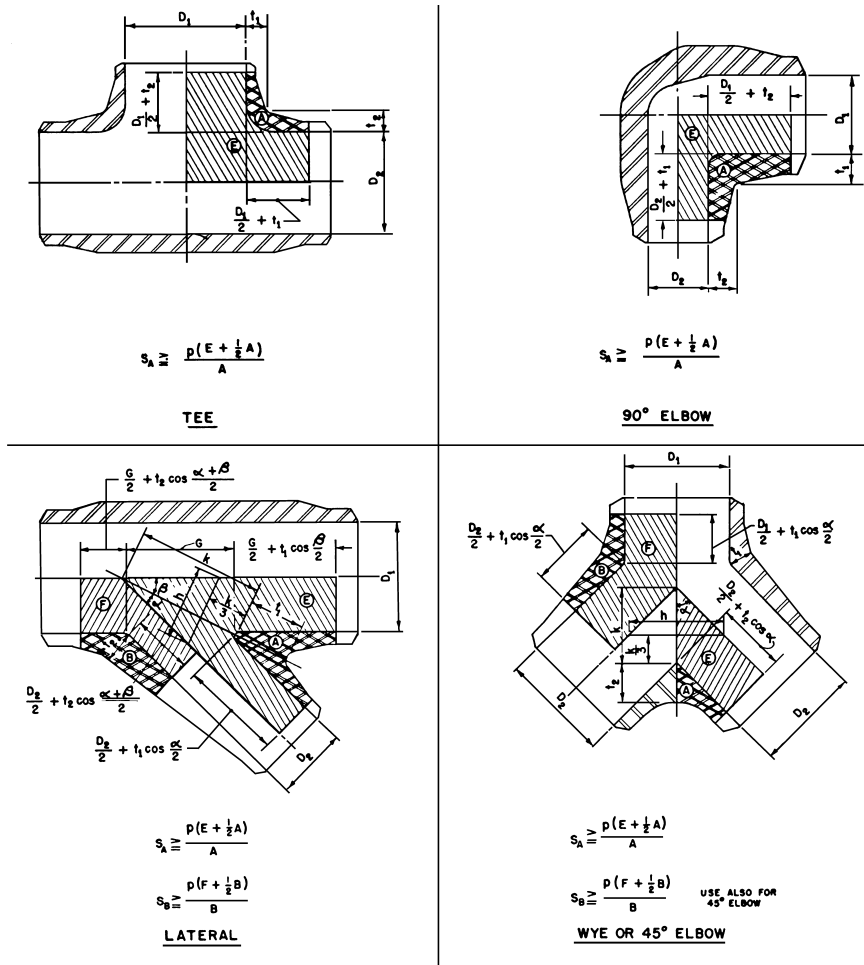


Figure 9-12. Fitting reinforcement. (Kellogg 1961.)

Another application of the pressure-area method is in analyzing conduit bifurcations, Fig. 9-14. The analysis of such structures is discussed by Swanson (Swanson, et al. 1955) and AISI (AISI 1981).

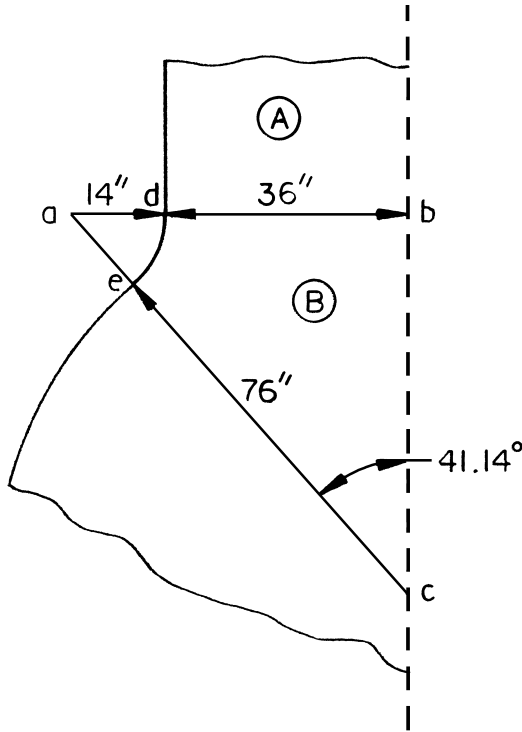


Figure 9-13. Cylindrical-to-spherical transition area.

Problems

9-3 Using the pressure-area method, show that the required thickness of the bellows expansion joint due to internal pressure can be approximated by

$$t = \frac{p(d + w)}{s(1.14 + 4w/q)}$$

where

$d \gg w$;

s = allowable stress;

p = internal pressure.

9-4 Calculate the area needed to reinforce the nozzle shown. Let $p = 800$ psi and $\sigma = 20$ ksi.

9-5 Calculate the thicknesses t_1 , t_2 , t_3 , t_4 , and area A of the section shown based on an average allowable stress of 18 ksi. Let $p = 250$ psi.

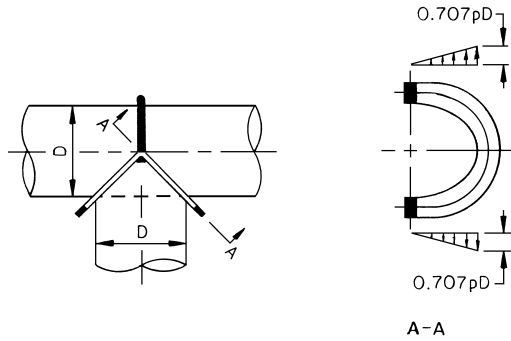


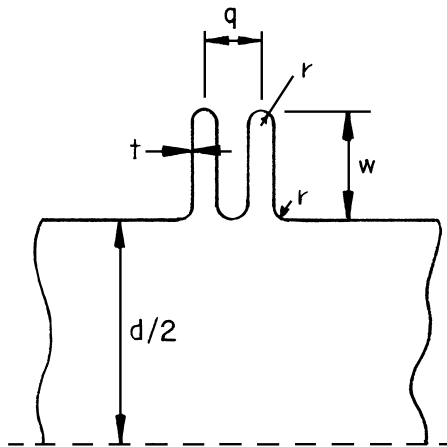
Figure 9-14. Conduit reinforcement. (Courtesy of Steel Plate Fabricators Association.)

9-6 The pipe elbow is subjected to a pressure of 4000 psi. Show that the average stress in the outer surface is equal to 10,170 psi and the stress in the inner surface is equal to 28,230 psi.

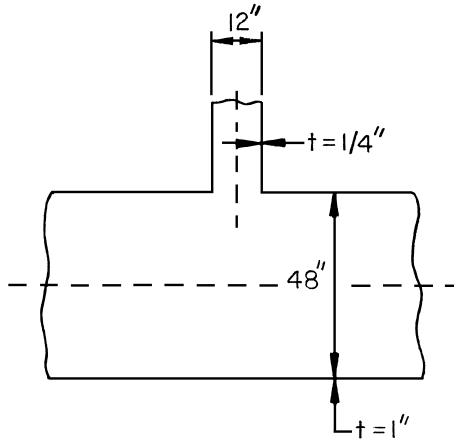
9-3 One-Sheet Hyperboloids

Many structures, Fig. 9-15, are shaped in the form of one-sheet hyperboloids. These include cooling towers in power plants as well as water storage tanks. The governing equation for the geometry of such structures is

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{b^2} = 1. \quad (9-1)$$



Prob. 9-3. Expansion joint.

**Prob. 9-4.** Nozzle in a shell.

This equation is plotted in Fig. 9-16. At any constant elevation $z = c$, the cross section is a circle. At the surface of this circle, such as $x = a$, Eq. (9-1) reduces to

$$\frac{y^2}{a^2} = \frac{z^2}{b^2} \quad (9-2)$$

or

$$z = \pm (b/a)y \quad (9-3)$$

which is the equation of a pair of straight lines that lie along the surface at that point.

Equation (9-1) can also be written in polar coordinates as

$$\frac{r^2}{a^2} - \frac{z^2}{b^2} = 1$$

or

$$r = \pm \frac{a}{b}(b^2 + z^2)^{1/2}. \quad (9-4)$$

This equation can also be written as

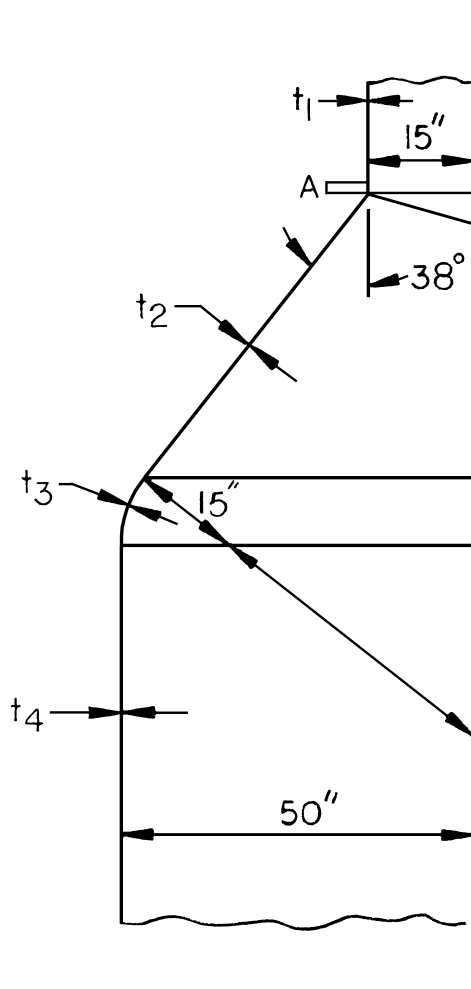
$$z = \pm \frac{b}{a}(r^2 - a^2)^{1/2}. \quad (9-5)$$

From Fig. 9-16 we get the relationship

$$\tan \phi = dz/dr.$$

Substituting Eq. (9-5) into this expression gives

$$\tan \phi = \pm \frac{b}{a} \left(\frac{r^2}{r^2 - a^2} \right)^{1/2}. \quad (9-6)$$

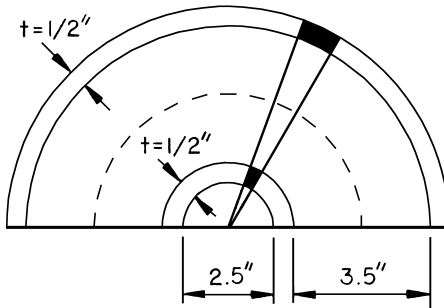


Prob. 9-5. Portion of a pressure vessel.

The equations for r_1 , r_2 , and r are expressed as (Flugge 1967)

$$r_1 = \frac{-a^2 b^2}{(a^2 \sin^2 \phi - b^2 \cos^2 \phi)^{3/2}} \quad (9-7)$$

$$r_2 = \frac{a^2}{(a^2 \sin^2 \phi - b^2 \cos^2 \phi)^{1/2}} \quad (9-8)$$



Prob. 9-6. Cross-section of a pipe elbow.

and

$$r = \frac{a^2 \sin \phi}{(a^2 \sin^2 \phi - b^2 \cos^2 \phi)^{1/2}} \quad (9-9)$$

A hyperboloid of a given shape can be expressed by Eq. (9-1) once a and b are established. The value of a is obtained at the equator where $z = 0$. The value of b is determined at a given elevation z . At any given elevation, the value of r is obtained from Eq. (9-4) and the value of ϕ is calculated from Eq. (9-6). At the same elevation, the values of r_1 and r_2 are obtained from Eqs. (9-7) and (9-8).



Figure 9-15. St. Louis Planetarium. (Courtesy of Jennifer Jawad.)

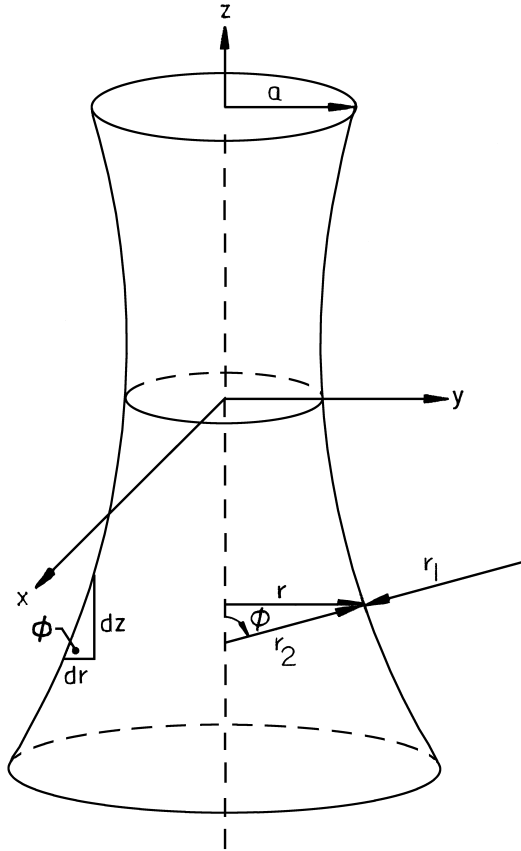


Figure 9-16. Hyperboloid.

The dead load of the hyperboloid at any given elevation can be determined from Fig. 9-17 as

$$W = \int 2\pi r \, dz = \int 2\pi r |r_1| \, d\phi \quad (9-10)$$

Substituting the values of r_1 and r from Eqs. (9-7) and (9-9) into this expression and defining Q and dQ (Kelkar and Sewell 1987) as

$$Q = \frac{\sqrt{a^2 + b^2}}{a} \cos \phi \quad (9-11)$$

$$dQ = -\frac{\sqrt{a^2 + b^2}}{a} \sin \phi \, d\phi \quad (9-12)$$

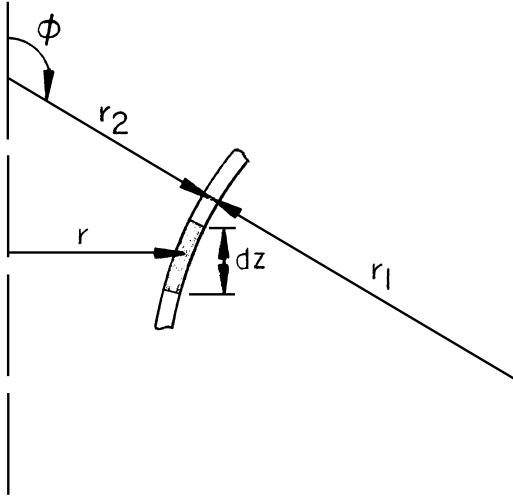


Figure 9-17. Section of hyperboloid.

gives

$$W = -\frac{\pi\gamma ab^2}{2\sqrt{a^2 + b^2}} \left[\frac{2Q}{1-Q^2} + \ln \frac{1+Q}{1-Q} \right]_{Q_0}^Q \quad (9-13)$$

where Q_0 is evaluated at the top of the tower and γ is the dead weight per unit surface area.

The direction of r_1 in Fig. 9-16 and Eq. (9-7) is opposite that shown in Fig. 8-4 due to the negative curvature of the hyperboloid. Accordingly, r_1 must be entered as a negative value in Eqs. (8-10) through (8-12) for the equilibrium of shells of revolution. Thus, Eq. (8-11) must be written as

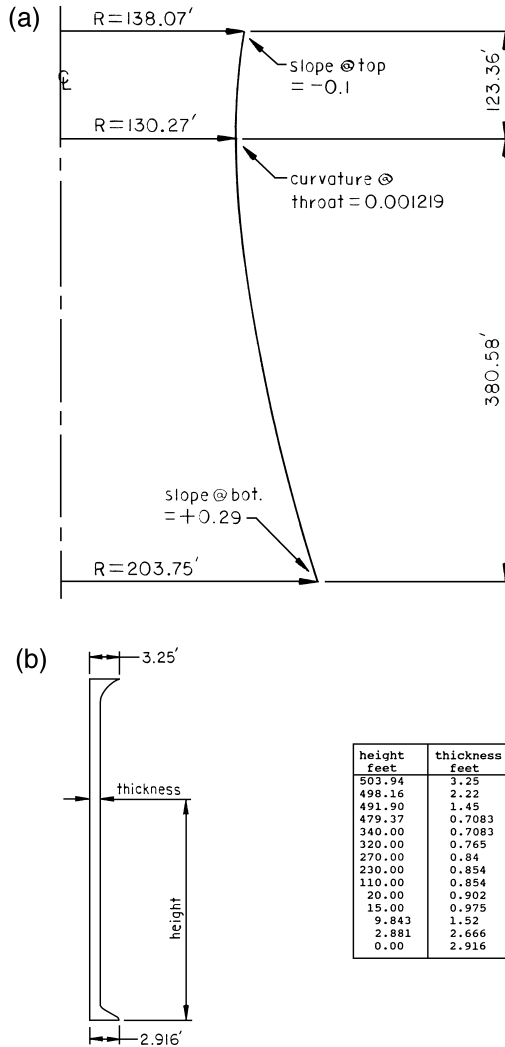
$$\frac{N_\theta}{r_2} - \frac{N_\phi}{|r_1|} = p_r. \quad (9-14)$$

For axisymmetric loads, Eq. (8-10) can be discarded and Eq. (8-12) defines the equilibrium of the shell at a given elevation as described in Section 9-1. Accordingly, the analysis of a one-sheet hyperboloid due to axisymmetric loads consists of establishing first the values of N_ϕ at various elevations in the structure. Then N_θ is calculated from Eq. (9-14) at these elevations.

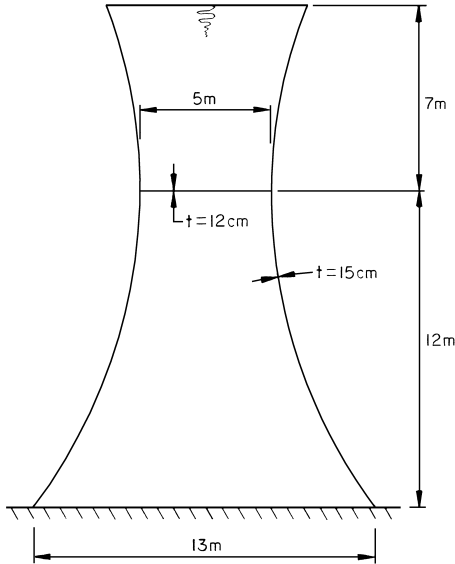
For nonsymmetric loads such as wind and earthquake forces, the solution becomes more complicated. References such as Flugge (Flugge 1967) and Gould (Gould 1988) discuss such loading conditions and their solutions.

Problems

9-7 Determine the forces N_ϕ and N_θ in the natural draft cooling tower shown in Fig. 9-7a due to dead weight. The thickness profile is shown in Fig. 9-7b. Calculate these forces at 20-ft increments in height. Assume the weight of the concrete as 150 pcf.



Prob. 9-7. Cooling tower. (Courtesy of Zurn Balcke-Durr, Tampa, FL.)



Prob. 9-8. Elevated water tower.

9-8 Determine the forces in the water tower. Disregard the dead weight of the tower and consider weight of the water only. Calculate the forces at 3-meter increments.

9-4 Deflection Due to Axisymmetric Loads

The deflection of a shell due to membrane forces caused by axisymmetric loads can be derived from Fig. 9-18. The change of length AB due to deformation is given by

$$\frac{dv}{d\phi} d\phi - w d\phi.$$

The strain is obtained by dividing this expression by the original length $r_1 d\phi$

$$\epsilon_\phi = \frac{1}{r_1} \frac{dv}{d\phi} - \frac{w}{r_1}. \quad (9-15)$$

The increase in radius r due to deformation, Fig. 9-19, is given by

$$v \cos \phi - w \sin \phi$$

or

$$\epsilon_\theta = \frac{1}{r} (v \cos \phi - w \sin \phi).$$

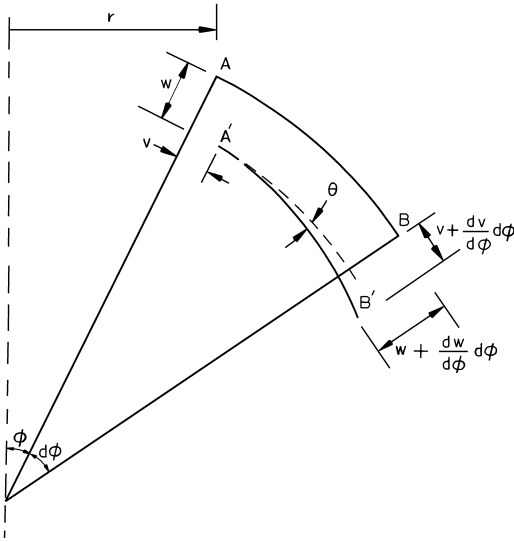


Figure 9-18. Deflection of shell.

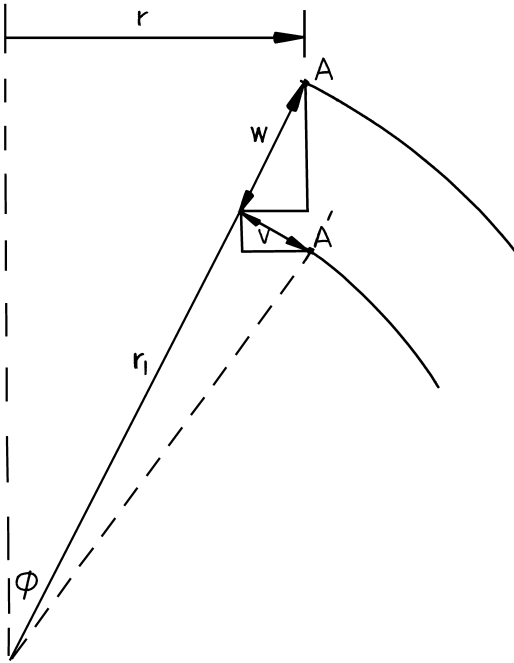


Figure 9-19. Radius increment due to deflection.

Substituting in this equation the value

$$r = r_2 \sin \phi$$

gives

$$\varepsilon_\theta = \frac{\nu}{r_2} \cot \phi - \frac{w}{r_2} \quad (9-16)$$

or

$$w = \nu \cot \phi - r_2 \varepsilon_\theta. \quad (9-17)$$

From Eqs. (9-15) and (9-16) we get

$$\frac{dv}{d\phi} - \nu \cot \phi = r_1 \varepsilon_\phi - r_1 \varepsilon_\theta. \quad (9-18)$$

From Eq. (1-13) for two-dimensional problems,

$$\left. \begin{aligned} \varepsilon_\phi &= \frac{1}{Et} (N_\phi - \mu N_\theta) \\ \varepsilon_\theta &= \frac{1}{Et} (N_\theta - \mu N_\phi) \end{aligned} \right] \quad (9-19)$$

Hence, Eq. (9-18) becomes

$$\frac{dv}{d\phi} - \nu \cot \phi = \frac{1}{Et} [N_\phi(r_1 + \mu r_2) - N_\theta(r_1 + \mu r_2)] \quad (9-20)$$

Let the righthand side of this equation be expressed as $g(\phi)$,

$$g(\phi) = \frac{1}{Et} [N_\phi(r_1 + \mu r_2) - N_\theta(r_1 + \mu r_2)] \quad (9-21)$$

and Eq. (9-6) is reduced to

$$\nu = \sin \phi \left(\int \frac{g(\phi)}{\sin \phi} + C \right). \quad (9-22)$$

In order to solve for the deflections in a structure due to a given loading condition, we first obtain N_ϕ and N_θ from Eqs. (8-12) and (8-11). We then calculate ν from Eqs. (9-21) and (9-22). The normal deflection, w , is then calculated from Eq. (9-17).

The rotation at any point is obtained from Figs. 9-18 and 9-19 as

$$\psi = \frac{1}{r_1} \left(\frac{dw}{d\phi} + \nu \right). \quad (9-23)$$

Example 9-3

Find the deflection at point A of the dome roof shown in Fig. 9-20 due to an internal pressure p . Let $\mu = 0.3$.

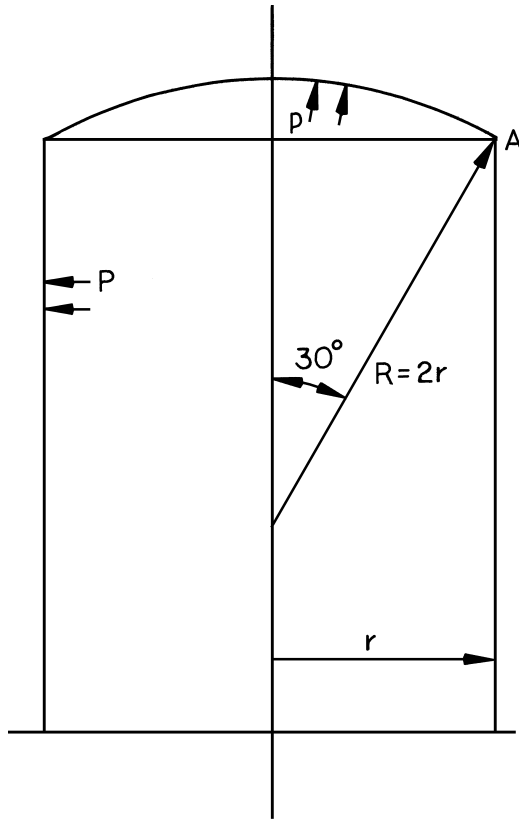


Figure 9-20. Flat bottom tank.

Solution

For a spherical roof with internal pressure,

$$r_1 = r_2 = R$$

$$p_r = p \quad \text{and} \quad p_\phi = 0.$$

The membrane forces are obtained from Eqs. (8-12) and (8-11) as

$$N_\phi = N_\theta = pR/2.$$

From Eq. (9-21),

$$\begin{aligned} g(\phi) &= \frac{1}{Et} [(pR/2)(R)(1 + \mu) - (pR/2)(R)(1 + \mu)] \\ &= 0 \end{aligned}$$

and Eq. (9-22) gives

$$v = C \sin \phi$$

at the point of support, $\phi = 30^\circ$, and the deflection $v = 0$. Hence, $C = 0$.

From Eq. (9-17)

$$w = -r_2 \varepsilon_\theta$$

and from Eq. (9-19),

$$w = \frac{-pR^2}{2Et} (1 - \mu). \quad (1)$$

At point A , the horizontal component of the deflection is given by

$$w_h = \frac{pR^2}{2Et} (1 - \mu) \sin \phi$$

Hence, with $\mu = 0.3$ and $\phi = 30^\circ$

$$w_h = \frac{0.175pR^2}{Et}.$$

If we substitute into this expression the quantity $R = 2r$ which is obtained from Fig. 9-20, we get

$$w_h = \frac{0.7pr^2}{Et}. \quad (2)$$

It will be seen in the next chapter that the deflection of a cylinder due to internal pressure is given by

$$w = \frac{pr^2}{Et} (1 - \mu/2)$$

or, for $\mu = 0.3$,

$$w = \frac{0.85pr^2}{Et} \quad (3)$$

A comparison of Eqs. (2) and (3) shows that at the roof-to-cylinder junction there is an offset in the calculated horizontal deflection. However, this offset does not exist in a real structure because of discontinuity forces that normally develop at the junction. These forces consist of local bending moments and shear forces. Although these forces eliminate the deflection offset, they do create high localized bending stresses. It turns out that these localized stresses are secondary in nature and can be discarded in the design of most structures as described in the next chapter.

Bending of Thin Cylindrical Shells due to Axisymmetric Loads

10-1 Basic Equations

The membrane forces discussed in the last two chapters are sufficient to resist many commonly encountered loading conditions. At locations where the deflection is restricted or there is a change in geometry such as cylindrical-to-spherical shell junction, the membrane theory is inadequate to maintain deflection and rotation compatibility between the shells as illustrated in Example 9-3. At these locations discontinuity forces are developed which result in bending and shear stresses in the shell. These discontinuity forces are localized over a small area of the shell and dissipate rapidly along the shell. Many structures such as missiles, Fig. 10-1, pressure vessels, and storage tanks are designed per the membrane theory and the total stress at discontinuities is determined from the membrane and bending theories. In this chapter the bending theory of cylindrical shells is developed and in Chapter 11 the bending theory of spherical and conical shells is discussed.

We begin the derivation of the bending of thin cylindrical shells by assuming the applied loads to be symmetric with respect to angle θ . A free-body diagram of an infinitesimal section of a cylindrical shell is shown in Fig. 10-2. The radius of the cylinder is designated as r . The applied loads p can vary in the x -direction only. At edges $x = 0$ and $x = dx$ the axial membrane force N_x , bending moments M_x , and shearing forces Q_x are axisymmetric. In the circumferential direction, only the hoop membrane force N_θ and bending moments M_θ are needed for equilibrium. There are no shearing forces, Q_θ , because the applied loads are symmetric in the circumferential direction. Summation of forces in the x -direction gives the first equation of equilibrium:

$$(N_x r \, d\theta) - \left(N_x + \frac{dN_x}{dx} dx \right) r \, d\theta = 0$$



Figure 10-1. The space shuttle Endeavour. (Courtesy of NASA.)

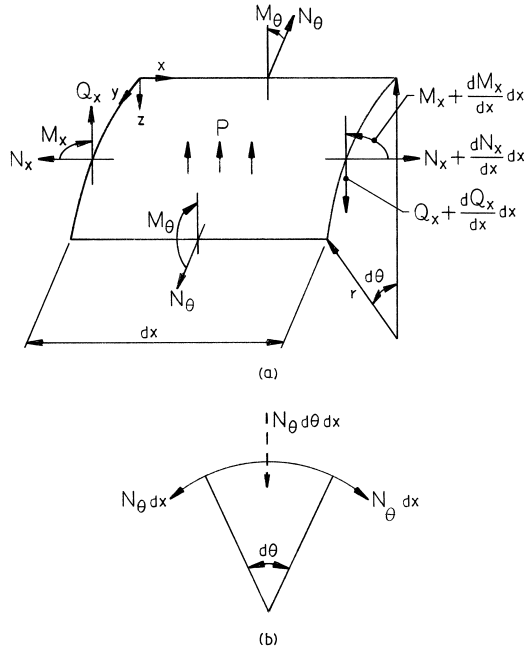


Figure 10-2. Infinitesimal section of a cylinder shell.

or

$$\frac{dN_x}{dx} r \, dx \, d\theta = 0. \quad (10-1)$$

This equation indicates that N_x must be a constant. We assume a cylinder with open ends and set $N_x = 0$. In Section 10-3, we will discuss the case where N_x is not zero.

Summation of forces in the z -direction gives the second equation of equilibrium:

$$Q_x r \, d\theta - \left(Q_x + \frac{dQ_x}{dx} dx \right) r \, d\theta - N_\theta \, dx \, d\theta + p r \, dx \, d\theta = 0$$

or

$$\frac{dQ_x}{dx} + \frac{N_\theta}{r} = p. \quad (10-2)$$

Summation of moments around the y -axis gives the third equation of equilibrium:

$$\begin{aligned} M_x r \, d\theta - \left(M_x + \frac{dM_x}{dx} dx \right) r \, d\theta + \left(Q_x + \frac{dQ_x}{dx} dx \right) (r \, d\theta) \, dx \\ - p r \, d\theta \frac{dx}{2} + 2N_\theta \frac{d\theta}{2} dx \frac{dx}{2} = 0. \end{aligned}$$

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After simplifying and deleting terms of higher order we get

$$\frac{dM_x}{dx} - Q_x = 0. \quad (10-3)$$

Eliminating Q_x from Eqs. (10-2) and (10-3) gives

$$\frac{d^2 M_x}{dx^2} + \frac{N_\theta}{r} = p. \quad (10-4)$$

Equation (10-4) contains two unknowns, N_θ and M_x . Both of these unknowns can be expressed in terms of deflection, w . Define axial strain as

$$\epsilon_x = \frac{du}{dx}. \quad (10-5)$$

The circumferential strain is obtained from Fig. 10-3 as

$$\epsilon_\theta = \frac{2\pi(r + \Delta r) - 2\pi r}{2\pi r}$$

$$\epsilon_\theta = \frac{\Delta r}{r}$$

or

$$\epsilon_\theta = \frac{-w}{r} \quad (10-6)$$

where w is the deflection and is taken as positive inwards. The stress-strain relationship given by Eq. (1-14) can be written in terms of force-strain

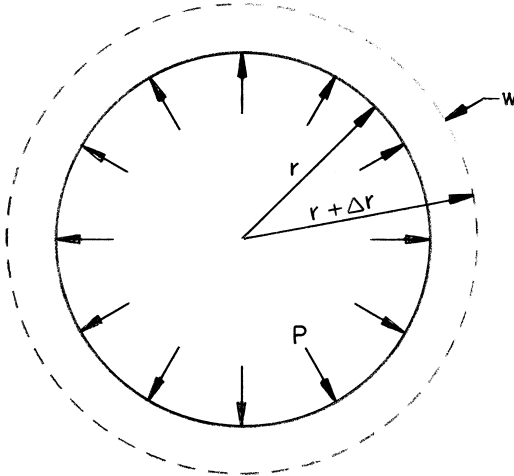


Figure 10-3. Circumferential deflection.

relationship as

$$\begin{bmatrix} N_x \\ N_\theta \end{bmatrix} = \frac{Et}{1-\mu^2} \begin{bmatrix} 1 & \mu \\ \mu & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_\theta \end{bmatrix} \quad (10-7)$$

Notice that the shearing strain, $\gamma_{x\theta}$, is zero in this case due to load symmetry in the θ -direction. Substituting Eqs. (10-5) and (10-6) into the first expression of Eq. (10-7) results in

$$N_x = \frac{Et}{1-\mu^2} \left(\frac{du}{dx} - \mu \frac{w}{r} \right).$$

Substituting into this expression the value $N_x = 0$ from Eq. (10-1), we get

$$\frac{du}{dx} = \mu \frac{w}{r}. \quad (10-8)$$

Similarly, the second term of Eq. (10-7) can be written as

$$N_\theta = \frac{Et}{1-\mu^2} \left(-\frac{w}{r} + \mu \frac{du}{dx} \right),$$

or upon inserting Eq. (10-8), it becomes

$$N_\theta = -\frac{Etw}{r}. \quad (10-9)$$

The basic moment-deflection relationships of Eq. (1-17) are also applicable to thin cylindrical shells. Referring to the two axes as x and θ rather than x and y , the first two expressions in Eq. (1-17) become

$$\begin{bmatrix} M_x \\ M_\theta \end{bmatrix} = -D \begin{bmatrix} 1 & \mu \\ \mu & 1 \end{bmatrix} \begin{bmatrix} \frac{d^2 w}{dx^2} \\ \frac{d^2 w}{d\theta^2} \end{bmatrix}. \quad (10-10)$$

It should be noted that x and θ in Eq. (10-10) are not in polar coordinates but redefined x - and y -axes. Polar transformation of Eq. (8-10) is given in Chapter 13.

The third expression, M_{xy} , in Eq. (1-17) vanishes because the rate of change of deflection with respect to θ is zero due to symmetry of applied loads. Also, due to symmetry with respect to θ , all derivatives with respect to θ vanish and the first expression in Eq. (10-10) reduces to

$$M_x = -D \frac{d^2 w}{dx^2} \quad (10-11)$$

and the second expression in Eq. (10-10) becomes

$$M_\theta = -\mu D \frac{d^2 w}{dx^2}. \quad (10-12)$$

From Eqs. (10-11) and (10-12) it can be concluded that

$$M_\theta = \mu M_x. \quad (10-13)$$

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Substituting Eqs. (10-9) and (10-11) into Eq. (10-4) gives

$$\frac{d^4 w}{dx^4} + \frac{Et}{Dr^2} w = -p(x)/D$$

which is the differential equation for the bending of cylindrical shells due to loads that are variable in the x -direction and uniformly distributed in the θ -direction. Defining

$$\beta^4 = \frac{Et}{4Dr^2} = \frac{3(1-\mu^2)}{r^2 t^2} \quad (10-14)$$

the differential equation becomes

$$\frac{d^4 w}{dx^4} + 4\beta^4 w = -p(x)/D \quad (10-15)$$

where p is a function of x .

Solution of Eq. (10-15) results in an expression for the deflection, w . The longitudinal and circumferential moments are then obtained from Eqs. (10-11) and (10-13), respectively. The circumferential membrane force, N_θ , is determined from Eq. (10-9).

One solution of Eq. (10-15) that is commonly used for long cylindrical shells is expressed as

$$w = e^{\beta x}(C_1 \cos \beta x + C_2 \sin \beta x) + e^{-\beta x}(C_3 \cos \beta x + C_4 \sin \beta x) + f(x). \quad (10-16)$$

Where $f(x)$ is the particular solution and C_1 to C_4 are constants that are evaluated from the boundary conditions.

A different solution of Eq. (10-15) that is commonly used for short cylindrical shells is expressed as

$$\begin{aligned} w = & C_1 \sin \beta x \sinh \beta x + C_2 \sin \beta x \cosh \beta x, \\ & + C_3 \cos \beta x \sinh \beta x + C_4 \cos \beta x \cosh \beta x. \end{aligned} \quad (10-17)$$

The procedure for establishing moments and forces in cylindrical shells well as defining long and short cylinders is discussed in the following sections.

For sign convention, M_x , Q_x , N_θ , and M_θ are all positive as shown in Fig. 10-2. The deflection, w , is positive inwards and the rotation, ψ , in the x -direction is positive in the direction of positive bending moments.

10-2 Long Cylindrical Shells

One application of Eq. (10-16) is of shear forces and bending moments applied at the edge of a cylindrical shell, Fig. 10-4. Referring to Eq. (10-16) for the deflection of a shell, we can set the function $f(x)$ to zero as there are no applied loads along the cylinder. Also, the deflection due to the term $e^{\beta x}$ in Eq. (10-16) tends to approach infinity as x gets larger. However, the deflection due to moments and forces applied at one end of an infinitely long cylinder

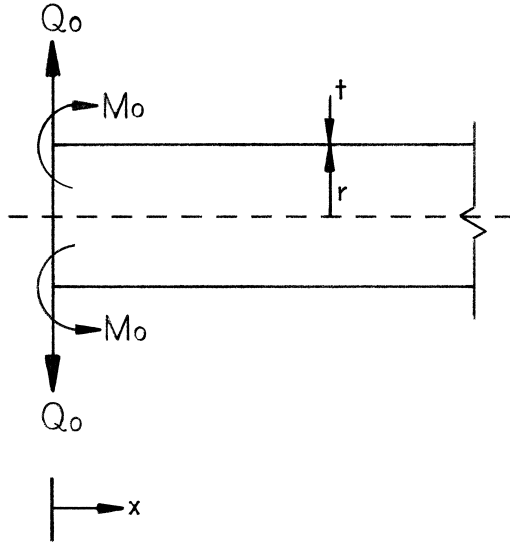


Figure 10-4. Shear and moment applied at edge of cylinder.

tend to dissipate as x gets larger. Thus, constants C_1 and C_2 must be set to zero and Eq. (10-16) becomes

$$w = e^{-\beta x}(C_3 \cos \beta x + C_4 \sin \beta x). \quad (10-18)$$

The boundary conditions for the infinitely long cylinder, Fig. 10-4, are obtained from Eq. (10-11) as

$$M_x|_{x=0} = M_0 = -D \frac{d^2 w}{dx^2} \Big|_{x=0}$$

and from Eq. (10-3) as

$$Q_x|_{x=0} = Q_0 = \frac{dM}{dx} \Big|_{x=0}.$$

Substituting Eq. (10-18) into the first boundary condition gives

$$C_4 = \frac{M_0}{2\beta^2 D}$$

and from the second boundary condition

$$C_3 = -\frac{1}{2\beta^3 D}(Q_0 + \beta M_0).$$

Hence, the deflection equation for the long cylinder shown in Fig. 10-4 is

$$w = \frac{e^{-\beta x}}{2\beta^3 D} [M_0(\sin \beta x - \cos \beta x) - Q_0 \cos \beta x]. \quad (10-19)$$

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By defining

$$A_{\beta x} = e^{-\beta x} (\cos \beta x + \sin \beta x) \quad (10-20)$$

$$B_{\beta x} = e^{-\beta x} (\cos \beta x - \sin \beta x) \quad (10-21)$$

$$C_{\beta x} = e^{-\beta x} \cos \beta x \quad (10-22)$$

$$D_{\beta x} = e^{-\beta x} \sin \beta x \quad (10-23)$$

the expression for the deflection and its derivative becomes

$$\text{deflection} = w_x = \frac{-1}{2\beta^3 D} (\beta M_o B_{\beta x} + Q_o C_{\beta x}) \quad (10-24)$$

$$\text{slope} = \psi_x = \frac{1}{2\beta^2 D} (2\beta M_o C_{\beta x} + Q_o A_{\beta x}) \quad (10-25)$$

$$\text{moment} = M_x = \frac{1}{2\beta} (2\beta M_o A_{\beta x} + 2Q_o D_{\beta x}) \quad (10-26)$$

$$\text{shear} = Q_x = -(2\beta M_o D_{\beta x} - Q_o B_{\beta x}). \quad (10-27)$$

The functions $A_{\beta x}$ through $D_{\beta x}$ are calculated in Table 10-1 for various values of βx .

Example 10-1

A long cylindrical shell is subjected to end moment M_o as shown in Fig. 10-5a. Plot the value of M_x from $\beta x = 0$ to $\beta x = 5.0$. Also determine the distance x at which the moment is about 1% of the original applied moment M_o .

Solution

From Eq. (10-26),

$$M_x = M_o A_{\beta x}.$$

The values of $A_{\beta x}$ are obtained from Eq. (10-20) and a plot of M_x is shown in Fig. 10-5b. From Eq. (10-26) and Table 10-1, the value of βx at which M_x is equal to about 1% of M_o is about 2.285. Hence,

$$\beta x = 2.285$$

or

$$x = 1.78\sqrt{rt} \quad \text{for} \quad \mu = 0.3.$$

The significance of the quantity $1.78\sqrt{rt}$ is apparent from Fig. 10-5b. It shows that a moment applied at the end of a long cylinder dissipates very rapidly as x increases and it reduces to about 1% of the original value at a distance of $1.78\sqrt{rt}$ from the edge. Many design codes such as the ASME VIII use a similar criterion

Table 10-1. Values of functions $A_{\beta x}$, $B_{\beta x}$, $C_{\beta x}$, $D_{\beta x}$

βx	$A_{\beta x}$	$B_{\beta x}$	$C_{\beta x}$	$D_{\beta x}$
0	1.0000	1.0000	1.0000	0.0000
0.05	0.9976	0.9025	0.9500	0.0475
0.10	0.9907	0.8100	0.9003	0.0903
0.15	0.9797	0.7224	0.8510	0.1286
0.20	0.9651	0.6398	0.8024	0.1627
0.30	0.9267	0.4888	0.7077	0.2189
0.40	0.8784	0.3564	0.6174	0.2610
0.50	0.8231	0.2415	0.5323	0.2908
0.60	0.7628	0.1431	0.4530	0.3099
0.80	0.6354	-0.0093	0.3131	0.3223
1.00	0.5083	-0.1108	0.1988	0.3096
1.20	0.3899	-0.1716	0.1091	0.2807
1.40	0.2849	-0.2011	0.0419	0.2430
1.60	0.1959	-0.2077	-0.0059	0.2018
1.80	0.1234	-0.1985	-0.0376	0.1610
2.00	0.0667	-0.1794	-0.0563	0.1231
2.50	-0.0166	-0.1149	-0.0658	0.0491
3.00	-0.0423	-0.0563	-0.0493	0.0070
3.5	-0.0389	-0.0177	-0.0283	-0.0106
4.0	-0.0258	0.0019	-0.0120	-0.0139
5.0	-0.0045	0.0084	0.0019	-0.0065
6.0	0.0017	0.0031	0.0024	-0.0007
7.0	0.0013	0.0001	0.0007	0.0006

for defining long cylinders where the forces applied at one end have a negligible effect at the other end.

Example 10-2

Determine the maximum stress in a long cylinder due to the radial load shown in Fig. 10-6a.

Solution

From the free-body diagram of Fig. 10-6b, the end load at point A is equal to $Q_o/2$. Also, from symmetry the slope is zero at point A . Thus,

$$\text{Slope due to } M_o + \text{slope due to } Q_o/2 = 0.$$

From Eq. (10-25)

$$\frac{M_o}{\beta D} - \frac{Q_o}{4\beta^2 D} = 0$$

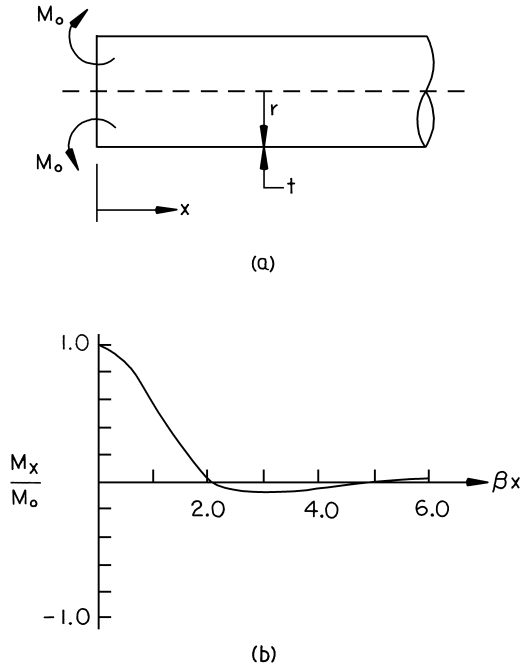


Figure 10-5. Long cylinder with edge moment M_o .

or

$$M_o = \frac{Q_o}{4\beta}.$$

The maximum longitudinal bending stress $= 6M_o/t^2 = \frac{3Q_o}{2\beta t^2}$

$$\text{Maximum deflection at point } A = \frac{-M_o}{2\beta^2 D} + \frac{Q_o}{4\beta^3 D} = \frac{Q_o}{8\beta^3 D}$$

The circumferential membrane force is obtained from Eq. (10-9) as

$$N_\theta = -\frac{Etw}{r} = -\frac{EtQ_o}{8r\beta^3 D}.$$

The circumferential bending moment is

$$M_\theta = \mu M_x$$

and the total maximum circumferential stress is

$$\sigma_\theta = -\frac{EQ_o}{8r\beta^3 D} - \frac{3\mu Q_o}{2\beta t^2}.$$

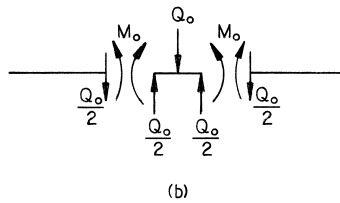
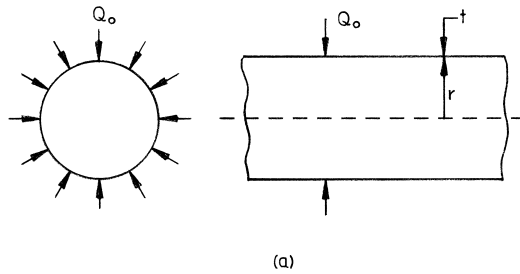


Figure 10-6. Long cylinder with a concentrated radial load Q_o .

Example 10-3

Determine the expression for stress in the long cylinder shown in Fig. 10-7a due to an internal pressure of 100 psi. The cylinder is supported by rigid bulkheads. Assume the longitudinal force N_x to be zero.

Solution

For a long cylindrical shell Eq. (10-16) can be written as

$$w = e^{-\beta x}(C_3 \cos \beta x + C_4 \sin \beta x) + f(x)$$

The particular solution for a constant pressure along the length of the cylinder can be expressed as

$$w_p = K.$$

Substituting this expression in Eq. (10-15) results in

$$w_p = -\frac{P}{4\beta^4 D}$$

which can also be expressed as

$$w_p = -\frac{pr^2}{Et}.$$

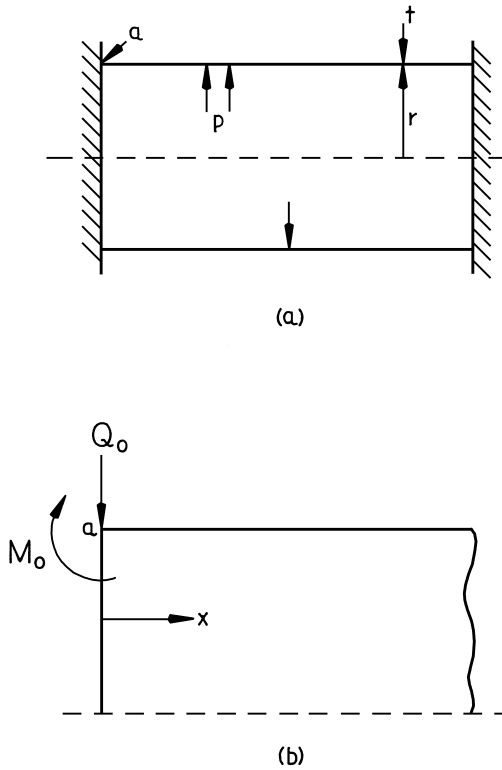


Figure 10-7. Long cylinder with fixed edges.

A free-body diagram of the discontinuity forces at the end is shown in Fig. 10-7b. The unknown moment and shear forces are assumed in a given direction as shown. A negative answer will indicate that the true direction is opposite the assumed one. The compatibility condition requires that the deflection at the edge due to pressure plus moment plus shear is equal to zero. Referring to Eq. (10-24) and Fig. 10-7,

$$\frac{-p}{4\beta^4 D} - \frac{M_o}{2\beta^2 D} + \frac{Q_o}{2\beta^3 D} = 0$$

or

$$M_o - \frac{Q_o}{\beta} = \frac{-p}{2\beta^2}. \quad (1)$$

Similarly the slope due to pressure plus moment plus shear is equal to zero at the edge. Hence, from Eq. (10-24) and Fig. 10-7,

$$0 + \frac{M_o}{\beta D} - \frac{Q_o}{2\beta^2 D} = 0$$

or

$$M_o - \frac{Q_o}{2\beta} = 0. \quad (2)$$

Solving Eqs. (1) and (2) gives

$$M_o = \frac{p}{2\beta^2} \quad \text{and} \quad Q_o = \frac{p}{\beta}.$$

Thus, the deflection is given by Eq. (10-24) and Fig. 10-7 as

$$w = \frac{p}{2\beta^4 D} \left(-\frac{1}{2} B_{\beta x} + C_{\beta x} \right) - \frac{p}{4\beta^4 D}.$$

At the bulkhead attachment, the circumferential membrane force N_θ is zero because the deflection is zero in accordance with Eq. (10-9). The axial bending moment is given by Eq. (10-11) as

$$M_x = -D \frac{d^2 w}{dx^2} = \frac{pe^{-\beta x}}{2\beta^2} (\sin \beta x - \cos \beta x).$$

The circumferential bending moment is given by Eq. (10-13) as

$$M_\theta = \mu M_x = \frac{\mu pe^{-\beta x}}{2\beta^2} (\sin \beta x - \cos \beta x).$$

The circumferential membrane force along the cylinder is given by Eq. (10-9) as

$$N_\theta = \frac{-Et}{r} \left(\frac{p}{2\beta^4 D} \right) (-B_{\beta x} + 2C_{\beta x} - 1).$$

longitudinal bending stress $\sigma_{Lb} = \pm 6M_x/t^2$

longitudinal membrane stress $\sigma_{Lm} = 0$

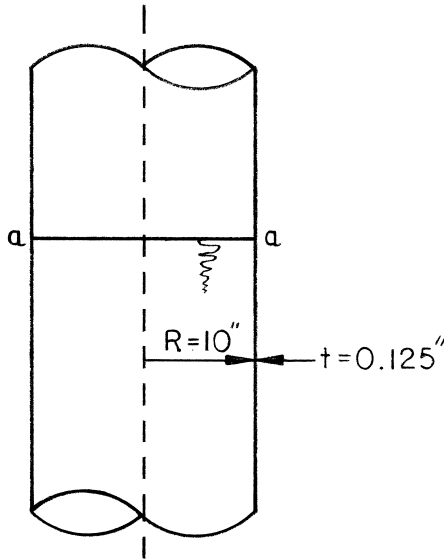
circumferential bending stress $\sigma_{\theta b} = \mu \sigma_{Lb}$

circumferential membrane stress $\sigma_{\theta m} = N_\theta/t$

Problems

10-1 In Example 10-1, assume a shearing force Q_o is applied at the end of the cylinder rather than M_o . Find the maximum longitudinal moment and its location from the edge. Let $\mu = 0.30$.

10-2 A cylindrical container is filled with a fluid to a level $a-a$. The metal temperature at a given time period is 400°F above section $a-a$ and 100°F below section $a-a$. Determine the discontinuity forces in the cylinder at section $a-a$. Let $\alpha = 6.5 \times 10^{-6}$ inch/inch/ $^\circ\text{F}$, $E = 30,000$ ksi, and $\mu = 0.30$. Disregard forces due to fluid pressure.



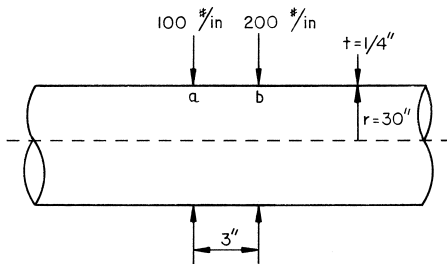
Prob. 10-2. Cylindrical shell partially filled with fluid.

10-3 Calculate the longitudinal bending stress at points a and b due to the applied loads shown. Let $E = 27,000$ ksi and $\mu = 0.32$.

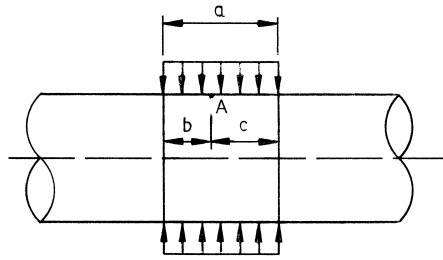
10-4 Show that for a uniform load over a small length, a , the deflection at point A is given by

$$w = \frac{pr^2}{2Et} (2 - e^{-\beta c} \cos \beta c - e^{-\beta b} \cos \beta b).$$

10-5 Find the expression for the bending moment in the water tank shown. Let $E = 20,000$ kgf/mm² and $\mu = 0.29$. *Hint:* Calculate first N_θ in terms of $p(x)$. Next use Eq. (10-9) to calculate w and dw/dx . Then use the fixed boundary condition at the bottom to calculate M_θ . M_x is then obtained from Eq. (10-26).



Prob. 10-3. Multiple radial loads on a cylinder.



Prob. 10-4. Distributed radial load on a cylinder.

10-3 Long Cylindrical Shells with End Loads

The radial deflection obtained from Eq. (10-18) is based on the assumption that the axial membrane force, N_x , in Eq. (10-1) is negligible. However, many applications involve pressure and hydrostatic loads that result in axial forces. The deflections and slopes due to these axial forces must be determined first and then the deflections and slopes due to edge effects described in the previous section are superimposed for a final solution. This procedure is illustrated here for a cylindrical shell with end closures and subjected to internal pressure.

Let a cylindrical shell, Fig. 10-8a, be subjected to internal pressure p . Then the circumferential, hoop, stress at a point away from the ends is obtained from Fig. 10-8b as

$$2\sigma_{\theta}tL = p(2r)L$$

or

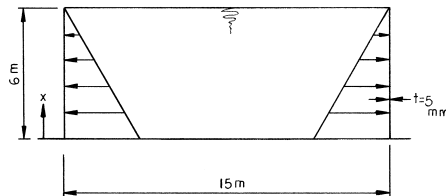
$$\sigma_{\theta} = \frac{pr}{t}. \quad (10-28)$$

Similarly the longitudinal stress is obtained from Fig. 10-8c as

$$\pi r^2 p = \sigma_x (2\pi r)t$$

or

$$\sigma_x = \frac{pr}{2t}. \quad (10-29)$$



Prob. 10-5. Water storage tank.

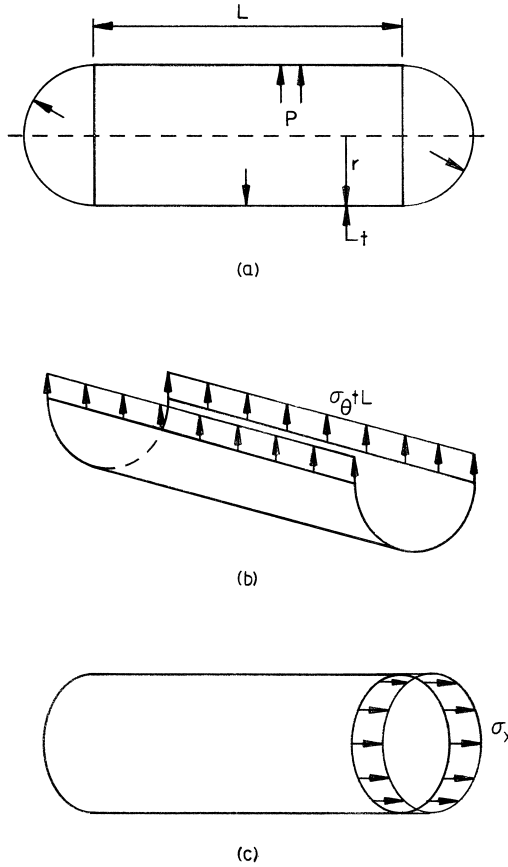


Figure 10-8. Cylindrical shell with end closures.

The maximum stresses given by Eqs. (10-28) and (10-29) for thin cylindrical shells subjected to internal pressure are valid as long as the shell is allowed to grow freely. Any restraints such as end closures and stiffening rings that prevent the shell from growing freely will result in bending moments and shear forces in the vicinity of the restraints. The magnitude of these moments and forces is determined subsequent to solving Eq. (10-18). The total stress will then be a summation of those obtained from Eqs. (10-28) and (10-29) plus those determined as a result of solving Eq. (10-18).

The circumferential and axial strains are obtained from Eqs. (1-14), with $\tau_{xy} = 0$, (10-28), and (10-29) as

$$\epsilon_{\theta} = \frac{pr}{Et}(1 - \mu/2) \quad (10-30)$$

$$\varepsilon_x = \frac{pr}{2Et}(1 - 2\mu). \quad (10-31)$$

The radial deflection is obtained from Fig. 10-3 as

$$\begin{aligned} \varepsilon_\theta &= \frac{2\pi(r + \Delta r) - 2\pi r}{2\pi r} \\ &= \frac{\Delta r}{r} = \frac{-w}{r} \end{aligned}$$

or

$$w = -r\varepsilon_\theta.$$

From this expression and Eq. (10-30), we get

$$w = -\frac{pr^2}{Et}(1 - \mu/2). \quad (10-32)$$

Example 10-4

A stiffening ring is placed around a cylinder at a distance removed from the ends as shown in Fig. 10-9a. The radius of the cylinder is 50 inches and its thickness is 0.25 inch. Also, the internal pressure = 100 psi, $E = 30,000$ ksi, and $\mu = 0.3$. Find the discontinuity stresses if (a) the ring is assumed to be infinitely rigid and (b) the ring is assumed to be 4 inches wide $\times$ 3/8 inch thick.

Solution

(a)

$$\beta = \sqrt[4]{\frac{3(1 - 0.3^2)}{50^2 \times 0.25^2}} = 0.36357$$

$$D = \frac{30,000,000 \times 0.25^3}{12(1 - 0.3^2)} = 42,925.82.$$

From Eq. (10-32),

$$w_p = \frac{100 \times 50^2}{30,000,000 \times 0.25}(1 - 0.3/2) = 0.0283 \text{ inch.}$$

From Fig. 10-9b it is seen that for an infinitely rigid ring, the deflection is zero. Also from symmetry, the slope is zero and Eqs. (10-24) and (10-25) give

$$w_M = \frac{M_o}{2\beta^2 D} \quad \theta_M = \frac{M_o}{\beta D}$$

$$w_Q = \frac{Q_o}{2\beta^3 D} \quad \theta_Q = \frac{Q_o}{2\beta^2 D}.$$

Total deflection at the ring attachment is equal to zero

$$0.0283 + \frac{M_o}{2\beta^2 D} - \frac{Q_o}{2\beta^3 D} = 0 \quad (1)$$

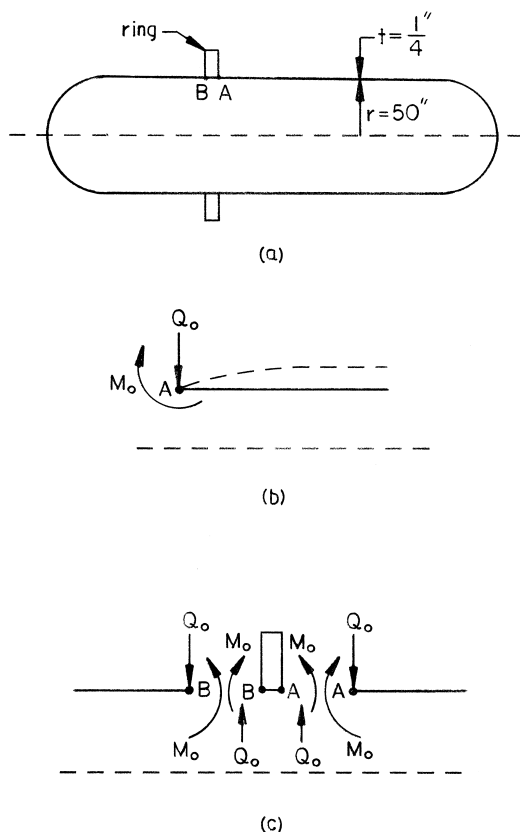


Figure 10-9. Pressure vessel with stiffening ring.

Similarly, the slope at the ring attachment is zero

$$\frac{M_o}{\beta D} - \frac{Q_o}{2\beta^2 D} = 0 \quad (2)$$

From Eq. (2),

$$Q_o = 2\beta M_o.$$

From Eq. (1),

$$M_o = 321.53 \text{ inch-lbs/inch.}$$

and

$$Q_o = 233.79 \text{ lbs/inch.}$$

Stress in the ring is zero because it is infinitely rigid.

longitudinal bending stress in cylinder $= 6M_o/t^2 = 30,870$ psi.

circumferential bending stress $= 0.3 \times 30,870 = 9260$ psi.

circumferential membrane stress at ring junction $= 0$ psi.

total longitudinal stress $= 30,870 + \frac{100 \times 50}{2 \times 0.25} = 40,870$ psi.

total circumferential stress $= 9260$ psi.

(b) From Fig. 10-9c, and symmetry, we can conclude that the shear and moment in the shell to the left of the ring are the same as the shear and moment to the right of the ring. Accordingly, we can solve only one unknown shear and one unknown moment value by taking the discontinuity forces of a shell on one side of the ring only. The deflection of the ring due to pressure can be ignored because the ring width is 16 times that of the shell. The deflection of the ring due to $2Q_o$ is given by

$$w_R = \frac{2Q_o r^2}{AE}.$$

Compatibility of the shell and ring deflections require that

deflection of shell = deflection of ring

or, from Fig. 10-9c,

$$\begin{aligned} &(\text{deflection due to } p - \text{deflection due to } Q + \text{deflection due to } M)|_{\text{shell}} \\ &= \text{deflection due to } 2Q_o|_{\text{ring}} \end{aligned}$$

$$\begin{aligned} \frac{pr^2}{Et}(1 - \mu/2) - \frac{Q_o}{2\beta^3 D} + \frac{M_o}{2\beta^2 D} &= \frac{2Q_o r^2}{AE} \\ M_o - 4.01Q_o &= -321.4. \end{aligned} \quad (1)$$

From symmetry, the rotation of the shell due to pressure plus Q_o , plus M_o must be set to zero.

$$\frac{Q_o}{2\beta^2 D} = \frac{M_o}{\beta D}$$

or

$$Q_o = 2\beta M_o. \quad (2)$$

Solving Eqs. (1) and (2) gives

$$Q_o = 122 \text{ lbs/inch}$$

$$M_o = 167.7 \text{ inch-lbs/inch.}$$

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Notice that this moment is about half of the moment in the case of an infinitely rigid ring.

$$\text{stress in ring} = \frac{2Q_0 r}{A} = 8130 \text{ psi.}$$

Maximum longitudinal stress in shell occurs at the ring attachment and is given by

$$\sigma_x = \frac{pr}{2t} + \frac{6M}{t^2} = 10,000 + 16,100 = 26,100 \text{ psi.}$$

Deflection of shell at ring junction is given by

$$\begin{aligned} w &= w_p - w_Q + w_M \\ w &= \frac{pr^2}{Et}(0.85) - \frac{Q_0}{2\beta^3 D} + \frac{M_0}{2\beta^2 D} \\ w &= \frac{406,120}{E}. \end{aligned}$$

The circumferential membrane force is

$$N_\theta = \frac{Etw}{(1-\mu^2)r} = 2231.4$$

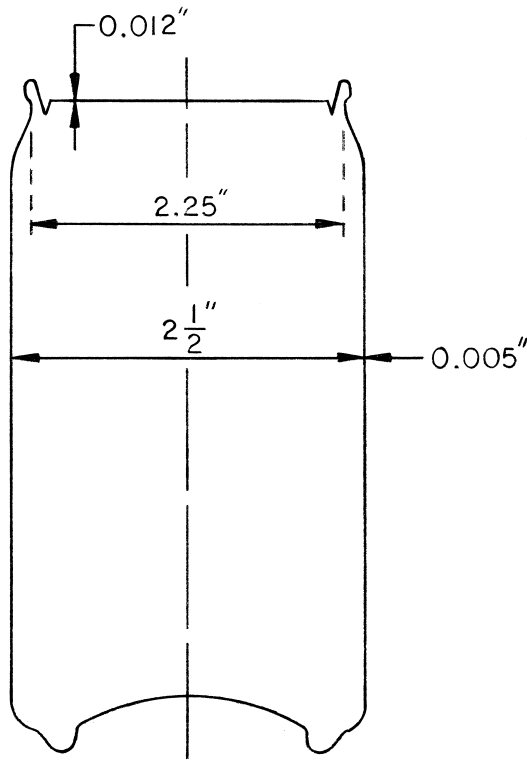
The circumferential bending moment is

$$\begin{aligned} M_\theta &= \mu M_x = 0.3 \times 167.7 = 50.3 \text{ inch-lbs/inch} \\ \sigma_\theta &= \frac{2231.4}{0.25} + \frac{6 \times 50.3}{0.25^2} = 8930 + 4830 = 13,760 \text{ psi.} \end{aligned}$$

Problems

10-6 The cross section of an aluminum beverage container be approximated as shown. Assume the top of the can to be a flat plate connected to an equivalent cylinder as shown by the shaded line. Determine the maximum permissible internal pressure. Let the shell allowable membrane stress = 30 ksi and the shell allowable bending plus membrane stress = 90 ksi. Let the allowable bending stress in the top plate = 45 ksi. Assume $E = 11,000$ ksi and $\mu = 0.33$.

Note: The pressure obtained from the assumptions made above is much lower than the actual pressure in the aluminum can. In the actual can there is an extension, or an expansion joint, between the top plate and the cylinder. Explain the effect of the expansion joint in increasing the permissible pressure. Also, determine the deflection of the top plate due to calculated pressure and calculate the membrane stress in the plate if it were treated as a spherical shell having a



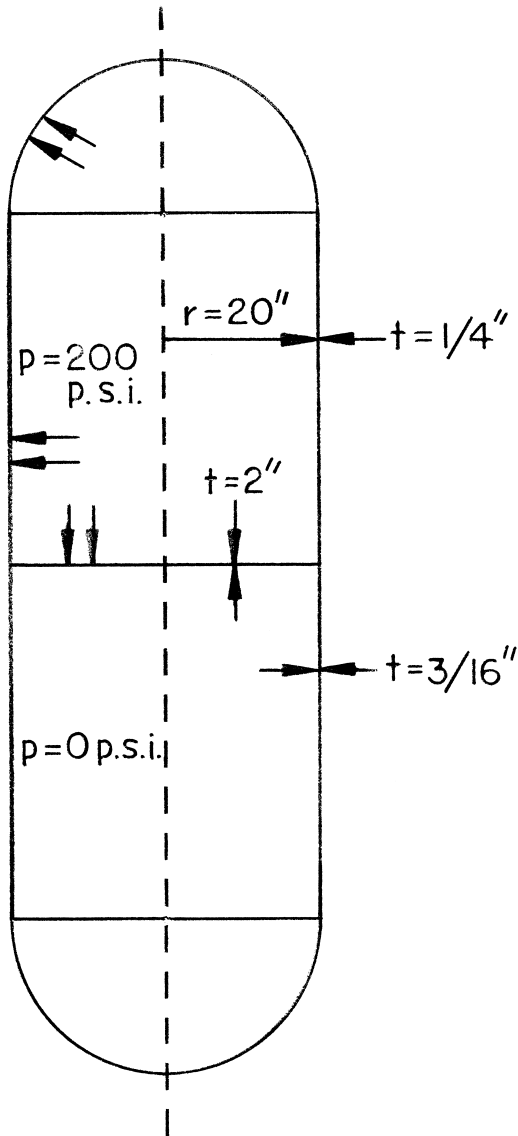
Prob. 10-6. Beverage container.

shape of the deflected plate. How does the membrane stress in the deflected plate compare with the bending stress of a flat plate?

10-7 Determine the maximum stress in the two cylindrical shells and in the 2-inch circular partition plate due to a pressure in the top compartment of 200 psi. Let $E = 25,000$ ksi and $\mu = 0.25$.

10-4 Short Cylindrical Shells

It was shown in the previous sections that the deflection due to applied edge shearing forces and bending moments dissipates rapidly as x increases and it becomes negligible at distances larger than $2.285/\beta$. This rapid reduction in deflection as x increased simplifies the solution of Eq. (10-16) by letting $C_1 = C_2 = 0$. When the length of the cylinder is less than about $2.285/\beta$, then C_1 and C_2 cannot be ignored and all four constants in Eq. (10-16) must be evaluated. Usually the alternate equation for the deflection, Eq. (10-17), results in a more



Prob. 10-7. Cylinder shell with partition plate.

convenient solution for short cylinders than Eq. (10-16). The calculations required in solving Eq. (10-17) are tedious because four constants are evaluated rather than two.

Example 10-5

Derive N_θ due to applied bending moment M_o at edge $x = 0$ for a short cylinder of length L .

Solution

The four boundary conditions are

At $x = 0$

$$\text{moment} = M_o = -D \left(\frac{d^2 w}{dx^2} \right)$$

$$\text{shear} = 0 = -D \left(\frac{d^3 w}{dx^3} \right).$$

At $x = L$

$$\text{moment} = 0 = -D \left(\frac{d^2 w}{dx^2} \right)$$

$$\text{shear} = 0 = -D \left(\frac{d^3 w}{dx^3} \right).$$

The second derivative of Eq. (10-17) is given by

$$\begin{aligned} \frac{d^2 w}{dx^2} = 2\beta^2 & (C_1 \cos \beta x \cosh \beta x + C_2 \cos \beta x \sinh \beta x \\ & - C_3 \sin \beta x \cosh \beta x - C_4 \sin \beta x \sinh \beta x). \end{aligned} \quad (1)$$

The third derivative of Eq. (10-17) is given by

$$\begin{aligned} \frac{d^3 w}{dx^3} = 2\beta^3 & [C_1 (\cos \beta x \sinh \beta x - \sin \beta x \cosh \beta x) \\ & + C_2 (\cos \beta x \cosh \beta x - \sin \beta x \sinh \beta x) \\ & - C_3 (\sin \beta x \sinh \beta x + \cos \beta x \cosh \beta x) \\ & - C_4 (\sin \beta x \cosh \beta x + \cos \beta x \sinh \beta x)]. \end{aligned} \quad (2)$$

Substituting Eq. (1) into the first boundary condition gives

$$C_1 = \frac{-M_o}{2D\beta^2}.$$

Substituting Eq. (2) into the second boundary condition gives

$$C_2 = C_3.$$

From the third and fourth boundary conditions we obtain

$$C_3 = \frac{M_o}{2D\beta^2} \left(\frac{\sin \beta L \cos \beta L + \sinh \beta L \cosh \beta L}{\sinh \beta L - \sin \beta L} \right)$$

and

$$C_4 = \frac{M_o}{2D\beta^2} \left(\frac{\sin \beta L + \sinh \beta L}{\sinh \beta L - \sin \beta L} \right).$$

From Eq. (10-9)

$$\begin{aligned} N_\theta &= \frac{Etw}{r} \\ &= \frac{Et}{r} (C_1 \sin \beta x \sinh \beta x + C_2 \sin \beta x \cosh \beta x \\ &\quad + C_3 \cos \beta x \sinh \beta x + C_4 \cos \beta x \cosh \beta x) \end{aligned}$$

Problems

10-8 Solve problem 10-4 if the length of the cylinder is 2.0 inches.

10-9 Solve problem 10-3 assuming the total length of the cylinder is 3.0 inches and the loads are applied at the edges.

10-5 Stress Due to Thermal Gradients in the Axial Direction

Thermal temperature gradients in cylindrical shells occur either along the axial length or through the thickness of cylinders. Thermal gradients through the thickness are discussed in Section 10-6. Stress in a cylindrical shell due to temperature gradients in the axial direction can be obtained by subdividing the cylinder into infinitesimal rings of length dx . The thermal expansion in each ring due to a change of temperature T_x within the ring is given by

$$w = \alpha r T_x$$

where

α is the coefficient of thermal expansion. Some values of α are shown in Table 2-1.

Since adjacent cylindrical rings cannot have a mismatch in the deflection due to temperature T_x at their interface, an assumed pressure p_x must be applied to eliminate the temperature deflection mismatch. Hence,

$$\frac{p_x r^2}{Et} = \alpha r T_x$$

and

$$\begin{aligned} p_x &= Et\alpha T_x / r \\ \sigma_\theta &= -p_x r / t = -E\alpha T_x. \end{aligned} \tag{10-33}$$

As the cylinder does not have any actual applied loads on it, the forces p_x must be eliminated by applying equal and opposite forces to the cylinder. Hence, Eq. (10-15) becomes

$$\frac{d^4 w}{dx^4} + 4\beta^4 w = \frac{Et\alpha T_x}{rD}. \quad (10-34)$$

The total stress in a cylinder due to axial thermal gradient distribution is obtained by adding the stresses obtained from Eqs. (10-33) and (10-34).

Example 10-6

The cylinder shown in Fig. 10-10a is initially at 0°F . The cylinder is heated as shown in the figure. Determine the thermal stresses in the cylinder. Let $\alpha = 6.5 \times 10^{-6}$ inch/inch/ $^\circ\text{F}$, $E = 30,000$ ksi, $L = 10$ ft, $t = 0.25$ inch, $r = 30$ inches, and $\mu = 0.3$. The cylinder is fixed at point A and free at point B .

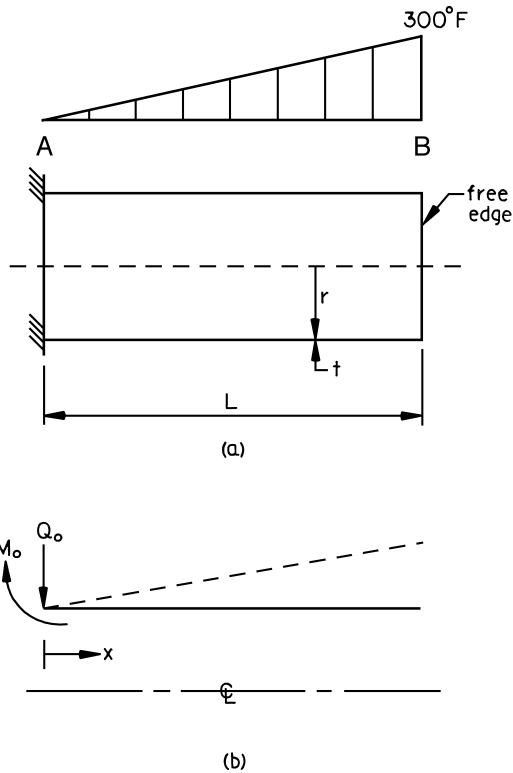


Figure 10-10. Temperature distribution along a cylinder.

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Solution

The temperature gradient is expressed as $T_x = 300 x/L$. The circumferential stress due to ring action is given by Eq. (10-33) as

$$\sigma_\theta = -E\alpha 300x/L \quad (1)$$

Equation (10-34) is written as

$$\frac{d^4 w}{dx^4} + 4\beta^4 w = \frac{Et\alpha}{rD} (300x/L).$$

A particular solution of this equation is taken as

$$w = C_1 x + C_2$$

Substituting this expression into the differential equation gives

$$C_1 = 300r\alpha/L \quad C_2 = 0$$

and

$$w = -r\alpha(300x/L).$$

From Eq. (10-9)

$$\sigma_\theta = N_\theta/t = Ew/r. \quad (2)$$

Adding Eqs. (1) and (2) gives

$$\sigma_\theta = 0$$

which indicates that the thermal stress in a cylinder due to linear axial thermal gradient is zero.

The slope, Fig. 10-10b, due to thermal gradient is

$$\theta = dw/dx = r\alpha(300/L)$$

At the fixed end, bending moments will occur due to the rotation θ caused by thermal gradients. The boundary conditions at the fixed edge are $w = 0$ and $\theta = 0$. From Eq. (10-24) and Fig. 10-10b,

$$0 = \beta M_o + Q_o.$$

From Eq. (10-25)

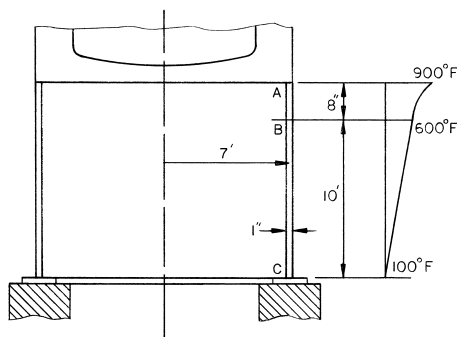
$$r\alpha(300/L) = \frac{1}{2\beta^2 D} (2\beta M_o + Q_o).$$

Solving these two equations gives

$$M_o = 2r\alpha(300)\beta D/L$$

$$= 19.63 \text{ inch-lbs/inch}$$

$$\sigma = 6M_o/t^2 = 6 \times 19.63/0.25^2 = 1880 \text{ psi.}$$



Prob. 10-10. Thermal gradient in a vessel support.

Problem

10-10 A pressure vessel in a refinery operates at 900°F . The ambient temperature is 100°F . The supporting cylinder is as shown. The temperature distribution in the cylinder is parabolic between points *A* and *B* and linear between points *B* and *C*. Find the thermal stress in the cylinder if it is assumed fixed at both ends. Let $\alpha = 7.0 \times 10^{-6}$ inch/inch/ $^{\circ}\text{F}$, $\mu = 0.30$, and $E = 29,000$ ksi.

10-6 Stress Due to Thermal Gradient in the Radial Direction

The governing equations for the thermal stress in a cylinder due to temperature gradient in the radial direction are obtained by combining various equations derived previously. It is assumed that all shearing strains are zero and that the strain due to temperature is given by αT where, α is the coefficient of thermal expansion and T is the temperature. It is also assumed that the axial strain ϵ_x is uniform and constant. Equation (1-13) can then be written as (Faupel and Fisher, 1981)

$$\epsilon_r = \frac{1}{E} [\sigma_r - \mu(\sigma_{\theta} + \sigma_z)] + \alpha T \quad (10-35)$$

$$\epsilon_{\theta} = \frac{1}{E} [\sigma_{\theta} - \mu(\sigma_z + \sigma_r)] + \alpha T \quad (10-36)$$

$$\epsilon_z = \frac{1}{E} [\sigma_z - \mu(\sigma_r + \sigma_{\theta})] + \alpha T \quad (10-37)$$

The circumferential strain is obtained from Eq. (10-6) as

$$\epsilon_{\theta} = w/r \quad (10-38)$$

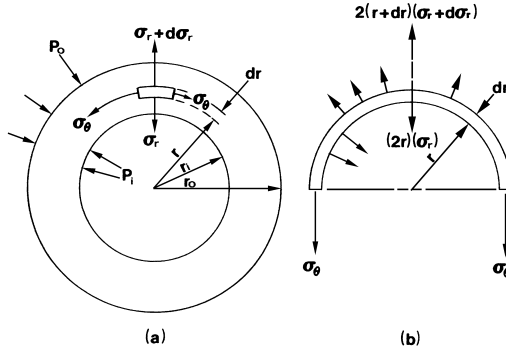


Figure 10-11. Circumferential and radial stress in a cylinder.

where outward radial deflection, w , is taken as positive. Radial strain is obtained from Fig. 10-3 as

$$\epsilon_r = dw/dr \quad (10-39)$$

Equations (10-38) and (10-39) are combined to give

$$\epsilon_r = \epsilon_\theta + r \left(\frac{d\epsilon_\theta}{dr} \right) \quad (10-40)$$

Substituting Eqs. (10-35) and (10-36) into Eq. (10-40) yields

$$\sigma_r = \sigma_\theta + r \left(\frac{d\sigma_r}{dr} \right) - \frac{\mu}{1-\mu} \left(\sigma_r + r \frac{d\sigma_r}{dr} - \sigma_\theta \right) + \left(\frac{E}{1-\mu^2} \right) \alpha (1+\mu) r \left(\frac{dT}{dr} \right) \quad (10-41)$$

The relationship between circumferential and radial stress in a thick cylinder is shown in Fig. 10-11a. Summation of vertical forces in Fig. 10-11b gives

$$\sigma_\theta - \sigma_r = r \left(\frac{d\sigma_r}{dr} \right) \quad (10-42)$$

Combining Eqs. (10-41) and (10-42) and solving for σ_r results in

$$r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (r^2 \sigma_r) \right] = \frac{-E}{1-\mu^2} \alpha (1+\mu) r \left(\frac{dT}{dr} \right) \quad (10-43)$$

Solving this differential equation and applying the boundary conditions

$$\sigma_r|_{r=r_i} = 0$$

$$\sigma_r|_{r=r_o} = 0$$

gives

$$\sigma_r = \frac{E\alpha}{1-\mu^2} \frac{1+\mu}{r^2} \left(\frac{r^2 - r_i^2}{r_o^2 - r_i^2} \int_{r_i}^{r_o} T r \, dr - \int_{r_i}^r T r \, dr \right) \quad (10-44)$$

From Eq. (10-42)

$$\sigma_\theta = \frac{E\alpha}{1-\mu^2} \frac{1+\mu}{r^2} \left(\frac{r^2 + r_i^2}{r_o^2 - r_i^2} \int_{r_i}^{r_o} Tr \, dr + \int_{r_i}^r Tr \, dr - Tr^2 \right) \quad (10-45)$$

and from Eq. (10-37) for a cylinder unrestrained in the z -direction ($\epsilon_z = \alpha T$),

$$\sigma_z = \frac{E\alpha}{1-\mu} \left(\frac{2}{r_o^2 - r_i^2} \int_{r_i}^{r_o} Tr \, dr - T \right) \quad (10-46)$$

From these three equations, some cases can be derived.

Case 1. Thin Shells with Linear Thermal Distribution

For thin vessels, a steady-state temperature condition produces linear thermal distribution through the thickness that can be expressed as

$$T = T_i \frac{r_o - r}{r_o - r_i} \quad (10-47)$$

where T_i = inside wall temperature relative to outside wall temperature. Substituting T into Eqs. (10-44) through (10-46) gives

$$\sigma_r = \frac{E\alpha T_i}{r^2(1-\mu)} \left[\frac{(r^2 - r_i^2)(2r_i + r_o)}{6(r_i + r_o)} - \frac{2(r^3 - r_i^3) - 3r_o(r^2 - r_i^2)}{6(r_i - r_o)} \right] \quad (10-48)$$

$$\sigma_\theta = \frac{E\alpha T_i}{r^2(1-\mu)} \left[\frac{(r^2 + r_i^2)(2r_i + r_o)}{6(r_i + r_o)} - \frac{2(r^3 + r_i^3) - 3r_o(r^2 + r_i^2)}{6(r_i - r_o)} \right] \quad (10-49)$$

$$\sigma_z = \frac{E\alpha T_i}{(1-\mu)} \left[\frac{2r_i + r_o}{3(r_i + r_o)} - \frac{r_o - r}{r_o - r_i} \right] \quad (10-50)$$

Figure 10-12 is a typical plot of σ_r , σ_θ , and σ_z . The plot indicates that σ_r is small compared with σ_θ and σ_z and for all practical purposes, σ_θ and σ_z are equal.

The maximum values of σ_θ and σ_z occur at the inner and outer surfaces. From Eqs. (10-48) and (10-49)

$$\sigma_\theta = \sigma_z = \begin{cases} \frac{-E\alpha T_i}{1-\mu} \left[\frac{2r_o + r_i}{3(r_o + r_i)} \right] & \text{for inside surface} \\ \frac{E\alpha T_i}{1-\mu} \left[\frac{r_o + 2r_i}{3(r_o + r_i)} \right] & \text{for outside surface} \end{cases} \quad (10-51)$$

For thin-wall vessels, Eq. (10-51) reduces to

$$\sigma_\theta = \sigma_z = \begin{cases} \frac{-E\alpha T_i}{2(1-\mu)} & \text{for inside surface} \\ \frac{E\alpha T_i}{2(1-\mu)} & \text{for outside surface} \end{cases} \quad (10-52)$$

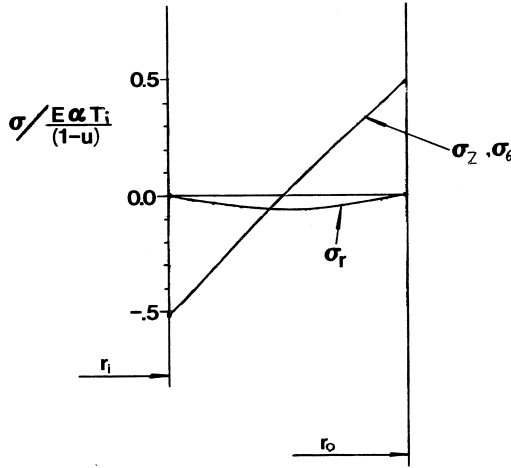


Figure 10-12. Thermal stress distribution in a wall of a cylinder.

Case 2. Thick Shells with Logarithmic Thermal Distribution

In thick vessels, a steady-state thermal condition gives rise to a logarithmic Temperature distribution (Burgreen, 1971) that can be expressed as

$$T = T_i \left(\frac{\ln r_o - \ln r}{\ln r_o - \ln r_i} \right) \quad (10-53)$$

Substitution of this expression in Eqs. (10-44) through (10-46) results in

$$\sigma_r = \frac{-E\alpha T_i}{2(1-\mu)\ln(r_o/r_i)} \left[\ln \frac{r_o}{r} + \frac{r_i^2}{r_o^2 - r_i^2} \left(1 - \frac{r_o^2}{r^2} \right) \ln \left(\frac{r_o}{r_i} \right) \right] \quad (10-54)$$

$$\sigma_\theta = \frac{E\alpha T_i}{2(1-\mu)\ln(r_o/r_i)} \left[1 - \ln \frac{r_o}{r} - \frac{r_i^2}{r_o^2 - r_i^2} \left(1 + \frac{r_o^2}{r^2} \right) \ln \left(\frac{r_o}{r_i} \right) \right] \quad (10-55)$$

$$\sigma_z = \frac{E\alpha T_i}{2(1-\mu)\ln(r_o/r_i)} \left[1 - 2\ln \frac{r_o}{r} - \frac{2r_i^2}{r_o^2 - r_i^2} \ln \left(\frac{r_o}{r_i} \right) \right] \quad (10-56)$$

Disregarding σ_r as being small compared with σ_θ and σ_z , Eqs. (10-54) and (10-56) have a maximum value of

$$\sigma_\theta = \sigma_z = \begin{cases} \frac{E\alpha T_i}{2(1-\mu)\ln(r_o/r_i)} \left[1 - \frac{2r_o^2}{r_o^2 - r_i^2} \ln \left(\frac{r_o}{r_i} \right) \right] & \text{for inside surface} \\ \frac{E\alpha T_i}{2(1-\mu)\ln(r_o/r_i)} \left[1 - \frac{2r_i^2}{r_o^2 - r_i^2} \ln \left(\frac{r_o}{r_i} \right) \right] & \text{for outside surface} \end{cases} \quad (10-57)$$

And for thin-wall cylinders, Eq. (10-57) reduces to

$$\sigma_\theta = \sigma_z = \begin{cases} \frac{-E\alpha T_i}{2(1-\mu)} & \text{for inside surface} \\ \frac{E\alpha T_i}{2(1-\mu)} & \text{for outside surface} \end{cases} \quad (10-58)$$

which are the same as those for the linear case.

Case 3. Thick Shells with Complex Thermal Distribution

In many applications, such as transient and upset conditions, the temperature distribution through the wall of a vessel cannot be represented by a mathematical expression. In this case a graphical solution can be obtained for the thermal stress. From Eq. (10-45)

$$\sigma_\theta = \frac{E\alpha}{1-\mu} \left[\frac{1 + (r_i/r)^2}{r_o^2 - r_i^2} \int_{r_i}^{r_o} Tr \, dr + \frac{1}{r^2} \int_{r_i}^r Tr \, dr - T \right] \quad (10-59)$$

For a cylinder where the thickness is small compared to the radius, the first expression in the brackets of Eq. (10-59) can be expressed as

$$\begin{aligned} \frac{1 + (r_i/r)^2}{r_o^2 - r_i^2} \int_{r_i}^{r_o} Tr \, dr &= \frac{2\pi \int_{r_i}^{r_o} Tr \, dr}{\pi(r_o^2 - r_i^2)} \\ &= \text{mean value of the temperature} \\ &\quad \text{distribution through the wall thickness} \\ &= T_m \end{aligned} \quad (10-60)$$

The second expression can be expressed as

$$\begin{aligned} \frac{1}{r^2} \int_{r_i}^r Tr \, dr &= \frac{2\pi \int_o^r Tr \, dr}{2\pi r^2} \\ &= \text{one-half the mean value of the temperature} \\ &\quad \text{distribution from the axis of the vessel to } r \end{aligned} \quad (10-61)$$

However, because the temperature distribution from the axis to r_i is zero, this latter expression for all practical purposes can be neglected. Hence, σ_θ can be expressed as

$$\sigma_\theta = \frac{E\alpha}{1-\mu} (T_m - T) \quad (10-62)$$

where

T_m = mean value of temperature distribution through the wall
 T = temperature at desired location

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From Eq. (10-46) it can be seen that σ_z can also be approximated by Eq. (10-62).

Example 10-7

A thin cylindrical vessel is heated from the outside such that the temperature distribution is as shown in Fig. 10-13. If $E = 29 \times 10^6$ psi, $\alpha = 9.3 \times 10^{-6}$ in./in.-°F, and $\mu = 0.28$, determine the maximum thermal stress (a) using Eq. (10-51) and (b) using Eq. (10-52).

Solution

(a): $T_i = 450 - 750 = -300^\circ\text{F}$. Hence at the inside surface

$$\sigma = \frac{-(29 \times 10^6)(9.3 \times 10^{-6})(-300)}{(1 - 0.28)} \left(\frac{2(13) + 11}{3(13 + 11)} \right)$$

$$= 57,750 \text{ psi}$$

and at the outside surface

$$\sigma = \frac{(29 \times 10^6)(9.3 \times 10^{-6})(-300)}{(1 - 0.28)} \left(\frac{13 + 2 \times 11}{3(13 + 11)} \right)$$

$$= 54,630 \text{ psi}$$

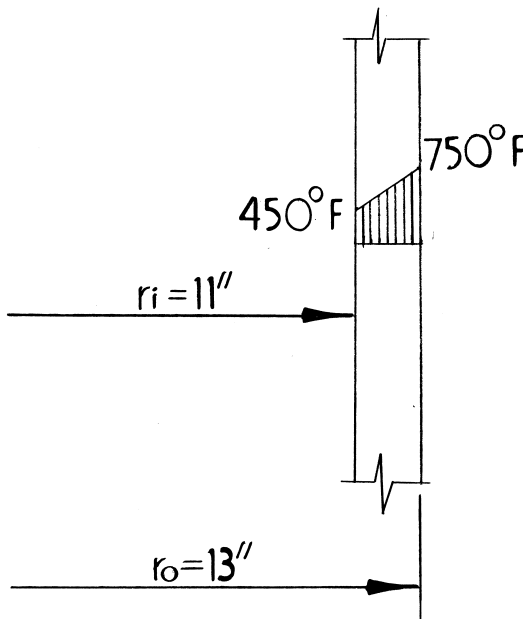


Figure 10-13. Temperature distribution in a thin-wall vessel.

(b): At the inside surface

$$\sigma = \frac{(-29 \times 10^6)(9.3 \times 10^{-6})(-300)}{2(1 - 0.28)}$$

$$= 56,200 \text{ psi}$$

and at the outside surface $\sigma = -56,200 \text{ psi}$.

Example 10-8

A pressure vessel operating at 400°F is subjected to a short excursion temperature of 700°F . At a given time, the temperature distribution in the wall is shown in Fig. 10-14. Find the maximum thermal stress at that instance. Let $\mu = 0.3$, $E = 30 \times 10^6 \text{ psi}$, and $\alpha = 6.0 \times 10^{-6} \text{ in./in.-}^\circ\text{F}$.

Solution

The maximum stress is determined from Eq. (10-62). The mean temperature is obtained from Fig. 10-14 and Table 10-2.

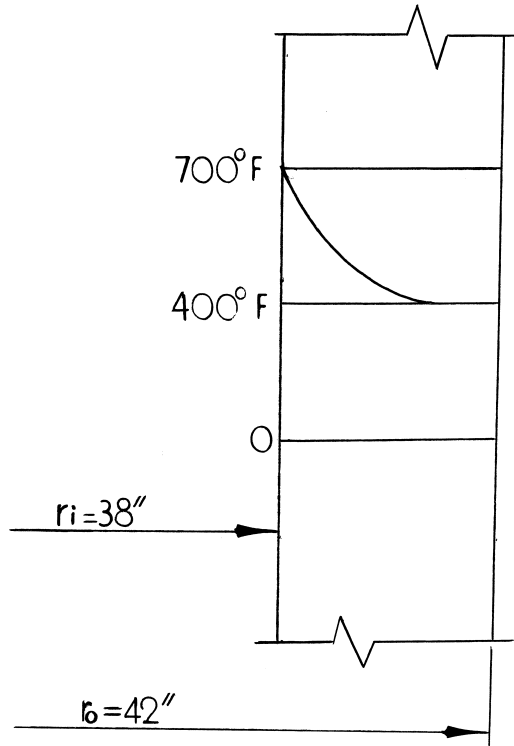


Figure 10-14. Temperature distribution in a thick-wall vessel.

Table 10-2. Mean Temperature

Location Across Thickness inch	Temperature °F	Area* inch-F
0.0	700	252
0.4	560	212
0.8	500	194
1.2	470	182
1.6	440	172
2.0	420	166
2.4	410	163
2.8	405	161
3.2	400	160
3.6	400	160
4.0	400	160
Sum		1822
Mean T	455.5	

*areas are approximated as rectangles and triangle.

From Eq. (10-62) at the inner surface,

$$\begin{aligned}\sigma &= \frac{(30 \times 10^6)(6.0 \times 10^{-6})}{1 - 0.3} (455.5 - 700) \\ &= -62,900 \text{ psi}\end{aligned}$$

and at the outer surface

$$\begin{aligned}\sigma &= \frac{(30 \times 10^6)(6.0 \times 10^{-6})}{1 - 0.3} (455.5 - 400) \\ &= 14,300 \text{ psi}\end{aligned}$$

It is of interest to note that the high stress occurs at the surface only. Thus at one-tenth of the thickness inside the surface, the stress is

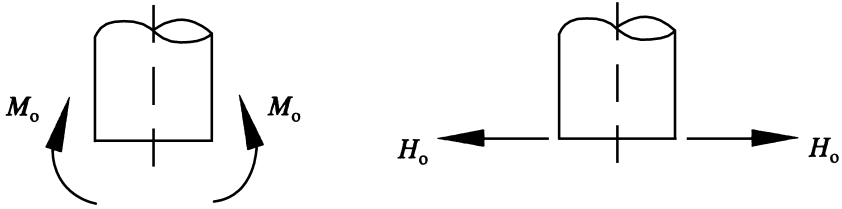
$$\begin{aligned}\sigma &= \frac{(30 \times 10^6)(6 \times 10^{-6})}{1 - 0.3} (455.5 - 560) \\ &= -26,900 \text{ psi}\end{aligned}$$

The high stress at the inner surface indicates that local yielding will occur.

10-7 Discontinuity Stresses

The design of various components in a shell structure subjected to axisymmetric loads consists of calculating the thickness of the main components first using the membrane theory and a given allowable stress. The forces due to various boundary conditions, such as those listed in Table 10-3, are then determined in accordance

Table 10-3. Various discontinuity functions

		
	Edge Functions	
w	$\frac{-M_o}{2\beta^2 D}$	$\frac{-H_o}{2\beta^3 D}$
θ	$\frac{-M_o}{\beta D}$	$\frac{H_o}{2\beta^2 D}$
M_x	M_o	0
N_θ	$2M_o\beta^2 r$	$2H_o\beta r$
Q_o	0	H_o
General Functions		
w	$\frac{-M_o}{2\beta^2 D} B_{\beta x}$	$\frac{-H_o}{2\beta^3 D} C_{\beta x}$
θ	$\frac{M_o}{\beta D} C_{\beta x}$	$\frac{H_o}{2\beta^2 D} A_{\beta x}$
M_x	$M_o A_{\beta x}$	$\frac{H_o}{\beta} D_{\beta x}$
N_θ	$2M_o\beta^2 r B_{\beta x}$	$2H_o\beta r C_{\beta x}$
Q_o	$2\beta M_o D_{\beta x}$	$H_o B_{\beta x}$

with the methods discussed in this chapter. In most cases, the magnitude of the discontinuity bending and membrane stresses at the junction is high. However, these high stresses are very local in nature and dissipate rapidly away from the junction as shown in the examples previously solved. Tests and experience have shown that these stresses are secondary in nature and are allowed to exceed the yield stress without affecting the structural integrity of the components.

Many Design Codes such as the ASME Pressure Vessel and Nuclear codes generally limit the secondary stresses at a junction to less than twice the yield stress at temperatures below the creep-rupture range. This stress level corresponds to approximately three times the allowable stress because the allowable stress is set at two-thirds the yield stress value. The justification for limiting the stress to twice the yield stress is best explained by referring to Fig. 10-15. The material stress-strain diagram is approximated by points ABO in Fig. 10-15. In the first loading cycle, the discontinuity stress, calculated elastically, at the junction increases from point A to B and then to C as the applied load is increased. The secondary stress is allowed to

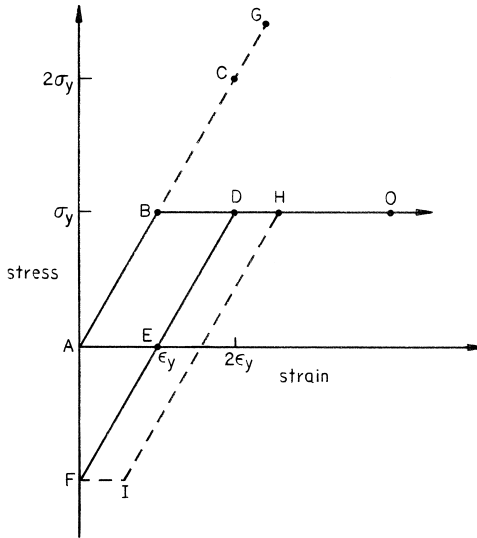


Figure 10-15. Stress-strain diagram.

approach twice the yield stress indicated by point *C*. This point corresponds to point *D* on the actual stress-strain diagram which is in the plastic range.

When the applied loads are reduced, the local discontinuity stress at point *D* is also reduced along the elastic line *DEF*. The high discontinuity stress at the junction is very localized in nature and the material around the localized area is still elastic. Thus, when the applied loads are reduced, the elastic material in the vicinity of the plastic region tends to return to its original zero strain and causes the much smaller volume of plastic material with high discontinuity stress to move from points *D* to *E* and then to *F*. Accordingly, at the end of the first cycle after the structure is loaded and then unloaded, the highly stressed discontinuity area that was stressed to twice the yield stress in tension is now stressed in compression to the yield stress value.

On subsequent loading cycles, the discontinuity stress is permitted to have a magnitude of twice the yield stress. However, the high stressed area which is now at point *F* moves to point *E* and then to point *D*. The high stress with a magnitude of twice the yield stress in the junction remains within the elastic limit on all subsequent loading cycles.

If the secondary discontinuity stress at the junction is allowed to exceed twice the yield stress such as point *G*, then for the first loading cycle the strain approaches point *H* on the actual stress-strain diagram. Downloading will follow the path from *H* to *I* and then to *F*. Hence, yielding of the junction will occur both in the up and down cycles. Subsequent cycles will continue the yielding process which results in incremental plastic deformation at the junction that could lead to premature fatigue failure.

Bending of Shells of Revolution Due to Axisymmetric Loads

11-1 Basic Equations

In Chapter 8 the basic equations of equilibrium for the membrane forces in shells of revolution due to axisymmetric loads were developed. Referring to Fig. 8-4, it was shown that the two governing equations of equilibrium are given by Eqs. (8-9) and (8-11) as

$$\frac{d}{d\phi}(rN_\phi) - r_1 N_\theta \cos \phi + p_\phi r r_1 = 0 \quad (11-1)$$

and

$$\frac{N_\phi}{r_1} + \frac{N_\theta}{r_2} = p_r. \quad (11-2)$$

In many applications such as at a junction of a spherical to cylindrical shell subjected to axisymmetric loads, bending moments and shear forces are developed at the junction in order to maintain equilibrium and compatibility between the two shells. The effect of these additional moments and shears, Fig. 11-1, on a shell of revolution is the subject of this chapter. The axisymmetric moments and shears at the two circumferential edges of the infinitesimal element are shown in Fig. 11-1. Circumferential bending moments, which are constant in the θ -direction for any given angle ϕ , are applied at the meridional edges of the element. The shearing forces at the meridional edges must be zero in order for the deflections, which must be symmetric in the θ -direction because the loads are axisymmetric, to be constant in the θ -direction for any given angle ϕ .

Summation of forces in Fig. 11-1 parallel to the tangent at the meridian results in

$$Q(r \, d\theta) \sin d\phi/2 + \left(Q + \frac{dQ}{d\phi} d\phi \right) \left(r + \frac{dr}{d\phi} d\phi \right) d\theta \sin d\phi/2 = 0.$$

Simplifying this equation gives

$$Qr \, d\phi \, d\theta = 0.$$

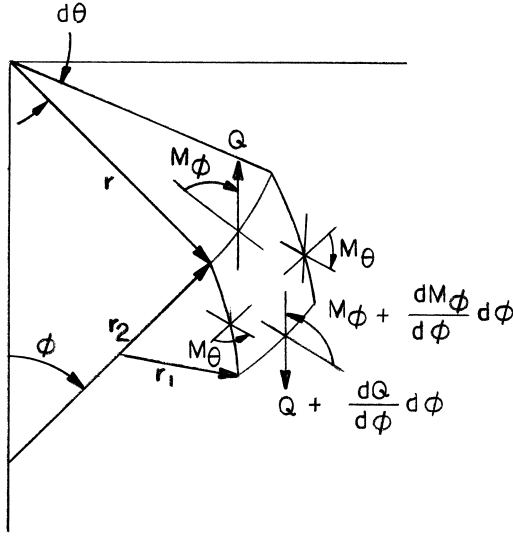


Figure 11-1. Bending and shear forces on an infinitesimal element.

Adding this expression to Eq. (11-1) gives

$$\frac{d}{d\phi}(rN_\phi) - r_1N_\theta \cos \phi - Qr + p_\phi rr_1 = 0. \quad (11-3)$$

Summation of forces perpendicular to the middle surface in Fig. 11-1 gives

$$-Q(r d\theta) \cos d\phi/2 + \left(Q + \frac{dQ}{d\phi} d\phi\right) \left(r + \frac{dr}{d\phi} d\phi\right) d\theta \cos d\phi/2 = 0.$$

Simplifying this expression and adding it to Eq. (11-2) gives

$$N_\phi r + N_\theta r_1 \sin \phi + \frac{d(Qr)}{d\phi} - p_r rr_1 = 0. \quad (11-4)$$

Summation of moments in the direction of a parallel circle gives

$$\frac{d}{d\phi}(M_\phi r) - M_\theta r_1 \cos \phi - Qr_1 r_2 \sin \phi = 0. \quad (11-5)$$

The second term involving M_θ in this equation is obtained from Fig. 11-2.

Equations (11-3) through (11-5) contain the five unknowns N_θ , N_ϕ , Q , M_θ , and M_ϕ . Accordingly, additional equations are needed and are obtained from the relationship between deflections and strains. The expressions for the meridional and circumferential strains were obtained in Chapter 9 as Eqs. (9-15) and (9-16) and are given by

$$\epsilon_\phi = \left(\frac{dv}{d\phi} - w\right) / r_1 \quad (11-6)$$

$$\epsilon_\theta = (v \cot \phi - w) / r_2. \quad (11-7)$$

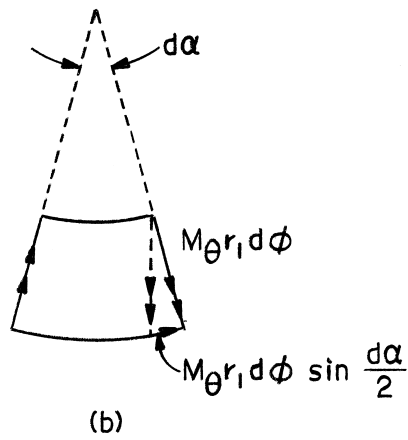
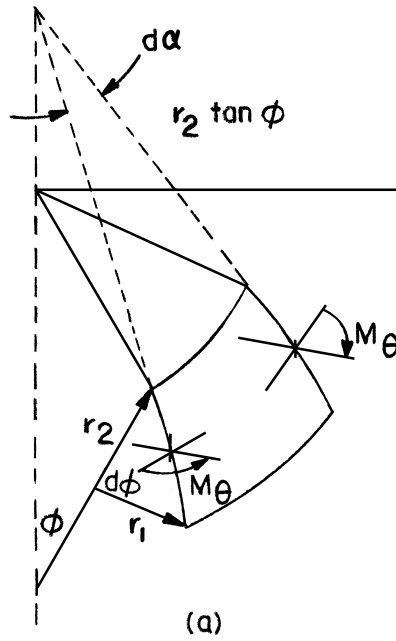


Figure 11-2. Components of M_θ .

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The expressions for N_θ and N_ϕ are obtained from Eq. (1-14) as

$$N_\phi = \frac{Et}{1 - \mu^2} (\varepsilon_\phi + \mu \varepsilon_\theta) \quad (11-8)$$

$$N_\theta = \frac{Et}{1 - \mu^2} (\varepsilon_\theta + \mu \varepsilon_\phi). \quad (11-9)$$

Substituting Eqs. (11-6) and (11-7) into Eqs. (11-8) and (11-9) gives

$$N_\phi = \frac{Et}{1 - \mu^2} \left[\frac{1}{r_1} \left(\frac{dv}{d\phi} - w \right) + \frac{\mu}{r_2} (v \cot \phi - w) \right] \quad (11-10)$$

$$N_\theta = \frac{Et}{1 - \mu^2} \left[\frac{1}{r_2} (v \cot \phi - w) + \frac{\mu}{r_1} \left(\frac{dv}{d\phi} - w \right) \right]. \quad (11-11)$$

The expression for change of curvature in the ϕ -direction is obtained from Fig. 11-3. The rotation α of point A in Fig. 11-3 is the summation of rotation α_1 due to deflection v and rotation α_2 due to deflection w , Fig. 11-4. From Fig. 11-4a,

$$\alpha_1 = v/r_1$$

and from Fig. 11-4b,

$$\alpha_2 = \frac{dw}{r_1 d\phi}.$$

Hence,

$$\alpha = \alpha_1 + \alpha_2 = \left(v + \frac{dw}{d\phi} \right) / r_1. \quad (11-12)$$

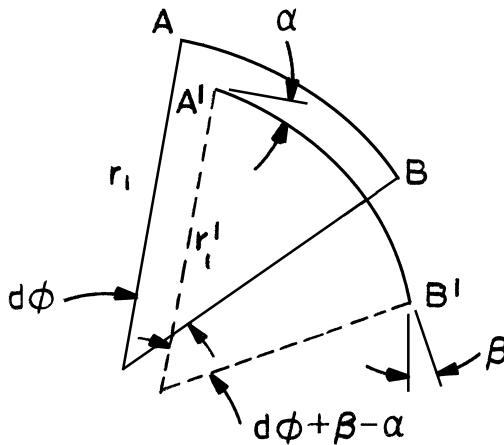
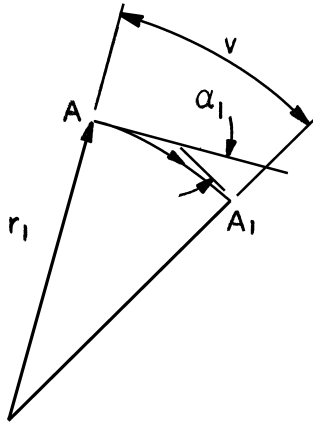
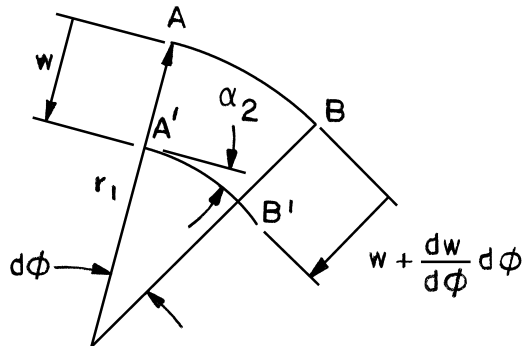


Figure 11-3. Deflection of a shell segment.



(a)



(b)

Figure 11-4. Rotation of a shell segment.

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Similarly the rotation β of point B is the summation of rotations due to deflection v and w . The rotation due to deflection v is expressed as

$$\left(v + \frac{dv}{d\phi} d\phi \right) / r_1$$

and the rotation due to w is

$$\frac{dw}{r_1 d\phi} + \frac{d}{d\phi} \left(\frac{dw}{r_1 d\phi} \right) d\phi.$$

Hence,

$$\beta = \left(v + \frac{dv}{d\phi} d\phi \right) / r_1 + \frac{dw}{r_1 d\phi} + \frac{d}{d\phi} \left(\frac{dw}{r_1 d\phi} \right) d\phi. \quad (11-13)$$

Due to rotation, the middle surface does not change in length. Thus, from Fig. 11-3

$$AB = A'B'$$

$$r_1 d\phi = r_1 (dQ + \beta - \alpha)$$

or,

$$\frac{1}{r_1'} = (d\phi + \beta - \alpha) / r_1 d\phi.$$

Change in curvature

$$\begin{aligned} \chi_\phi &= \frac{1}{r_1'} - \frac{1}{r_1} \\ &= \frac{(\beta - \alpha)}{r_1 d\phi} \\ \chi_\phi &= \frac{d}{r_1 d\phi} \left(\frac{v}{r_1} + \frac{dw}{r_1 d\phi} \right). \end{aligned} \quad (11-14)$$

The change of curvature in the θ -direction is obtained from Fig. 11-5 which shows the rotation of side AB due to the deformation of element $ABCD$. The original length AB is given by

$$AB = r_2 \sin \phi d\theta.$$

After rotation, AB is expressed by

$$AB = r_2' \sin (\phi + \alpha) d\theta.$$

Equating these two expressions and assuming small angle rotation yields

$$\frac{1}{r_2'} = \frac{1}{r_2} (1 + \alpha \cot \phi)$$

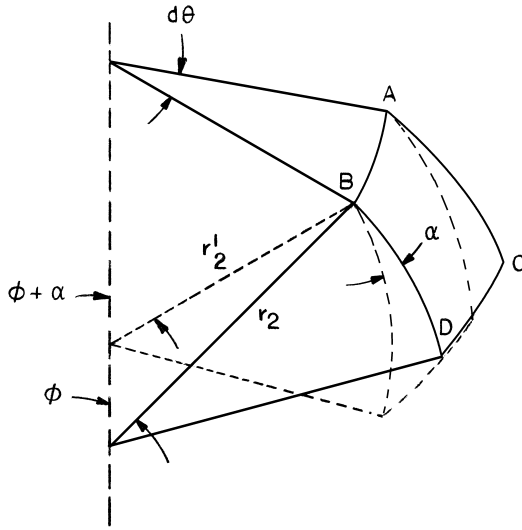


Figure 11-5. Rotation of a shell element in the θ - direction.

and from

$$\chi_{\theta} = \frac{1}{r'_2} - \frac{1}{r_2}$$

we get

$$\chi_{\theta} = \frac{\alpha \cot \phi}{r_2}.$$

Substituting Eq. (11-12) into this expression gives

$$\chi_{\theta} = \frac{1}{r_2} \left(\frac{v}{r_1} + \frac{dw}{r_1 d\phi} \right) \cot \phi. \quad (11-15)$$

The relationship between moment and rotation is obtained from Eqs. (1-6), (1-7), and (1-17) as

$$M_{\phi} = D(\chi_{\phi} + \mu\chi_{\theta}) \quad (11-16)$$

$$M_{\theta} = D(\chi_{\theta} + \mu\chi_{\phi}). \quad (11-17)$$

Substituting Eqs. (11-14) and (11-15) into these two expressions gives

$$M_{\phi} = D \left(\frac{1}{r_1} \frac{d\alpha}{d\phi} + \frac{\mu}{r_2} \alpha \cot \phi \right) \quad (11-18)$$

$$M_{\theta} = D \left(\frac{1}{r_2} \alpha \cot \phi + \frac{\mu}{r_1} \frac{d\alpha}{d\phi} \right). \quad (11-19)$$

The eight equations (11-3), (11-4), (11-5), (11-10), (11-11), (11-12), (11-18), and (11-19) contain eight unknowns. They are N_{θ} , N_{ϕ} , Q , M_{ϕ} , M_{θ} , v , w , and α . Solution of these equations is discussed next.

11-2 Spherical Shells

The forces and moments throughout a spherical shell due to edge shears and moments will be derived in this section. For spherical shells, Fig. 11-1, the radii r_1 and r_2 are equal to R . Also, the pressures p_r and p_ϕ are set to zero for the case of applied edge loads and moments. The eight pertinent equations derived in Section 11-1 can now be reduced to two differential equations. The first equation is obtained by substituting the moment Eqs. (11-18) and (11-19) into Eq. (11-5). This gives

$$\frac{d^2\alpha}{d\phi^2} + \frac{d\alpha}{d\phi} \cot \phi - \alpha(\cot^2 \phi + \mu) = -\frac{QR^2}{D}. \quad (11-20)$$

The second differential equation is more cumbersome to derive. We start by substituting Eq. (11-3) into Eq. (11-4) to delete N_θ . Then integrating the resultant equation with respect to ϕ gives

$$N_\phi = -Q \cot \phi. \quad (11-21)$$

Substituting this expression into Eq. (11-4) yields

$$N_\theta = -\frac{dQ}{d\phi}. \quad (11-22)$$

From Eqs. (11-10) and (11-11) we get

$$\frac{dv}{d\phi} - w = R(N_\phi - \mu N_\theta)/Et \quad (11-23)$$

and

$$v \cot \phi - w = R(N_\theta - \mu N_\phi)/Et. \quad (11-24)$$

Combining Eqs. (11-23) and (11-24) results in

$$\frac{dv}{d\phi} - v \cot \phi = R(1 + \mu)(N_\phi - N_\theta)/Et. \quad (11-25)$$

Differentiating Eq. (11-24) and combining it with Eq. (11-25) gives the expression

$$v + \frac{dw}{d\phi} = \frac{R}{Et} \left[(1 + \mu)(N_\phi - N_\theta) - \frac{dN_\theta}{d\phi} - R \frac{dN_\phi}{d\phi} \right].$$

Substituting Eqs. (11-21) and (11-22) into this expression results in

$$\frac{d^2Q}{d\phi^2} + \frac{dQ}{d\phi} \cot \phi - Q(\cot^2 \phi - \mu) = Et\alpha. \quad (11-26)$$

Equations (11-20) and (11-26) must be solved simultaneously to determine Q and α . The exact solution of these two equations is too cumbersome to use for most practical problems and is beyond the scope of this book. Timoshenko (Timoshenko and Woinowsky-Krieger 1959) showed that a rigorous solution of

Eqs. (11-20) and (11-26) results in expressions for α and Q that contain the terms $e^{\lambda\phi}$ and $e^{-\lambda\phi}$ where λ is a function of R/t . These terms have a large numerical value for thin shells with large R/t ratios. Substitution of these terms into Eqs. (11-20) and (11-26) for shells with large ϕ angles results in two equations with substantially larger numerical values for the higher derivatives $d^2Q/d\phi^2$ and $d^2\alpha/d\phi^2$ compared to the other terms in the equations. Hence, a reasonable approximation of Eqs. (11-20) and (11-26) is obtained by rewriting them as

$$\frac{d^2\alpha}{d\phi^2} = -\frac{QR^2}{D} \quad (11-27)$$

and

$$\frac{d^2Q}{d\phi^2} = Et\alpha. \quad (11-28)$$

Substituting Eq. (11-27) into Eq. (11-28) gives

$$\frac{d^4Q}{d\phi^4} + 4\lambda^4Q = 0 \quad (11-29)$$

where

$$\left. \begin{aligned} \lambda^4 &= EtR^2/4D \\ &= 3(1 - \mu^2)(R/t)^2. \end{aligned} \right] \quad (11-30)$$

The solution of Eq. (11-29) is similar to that obtained for cylindrical shells and is given by either

$$Q = e^{\lambda\phi}(C_1 \sin \lambda\phi + C_2 \cos \lambda\phi) + e^{-\lambda\phi}(C_3 \sin \lambda\phi + C_4 \cos \lambda\phi) \quad (11-31)$$

or

$$\begin{aligned} Q &= K_1 \sin \lambda\phi \sinh \lambda\phi + K_2 \sin \lambda\phi \cosh \lambda\phi + K_3 \cos \lambda\phi \sinh \lambda\phi \\ &\quad + K_4 \cos \lambda\phi \cosh \lambda\phi. \end{aligned} \quad (11-32)$$

For continuous spherical shells without holes and subjected to edge forces, the constants C_3 and C_4 in Eq. (11-31) must be set to zero in order for Q to diminish as ϕ gets smaller. Hence, Eq. (11-31) becomes

$$Q = e^{\lambda\phi}(C_1 \sin \lambda\phi + C_2 \cos \lambda\phi). \quad (11-33)$$

This equation can be written in a more compact form by substituting for ϕ the quantity $(\phi_0 - \zeta)$ shown in Fig. 11-6. The new equation with redefined constants is

$$Q = C_1 e^{-\lambda\zeta} \sin (\lambda\zeta - C_2). \quad (11-34)$$

After obtaining Q , other forces and moments can be determined. Thus, from Eq. (11-21), we obtain the longitudinal membrane force

$$N_\phi = -C_1 e^{-\lambda\zeta} \sin (\lambda\zeta + C_2) \cot (\phi_0 - \zeta). \quad (11-35)$$

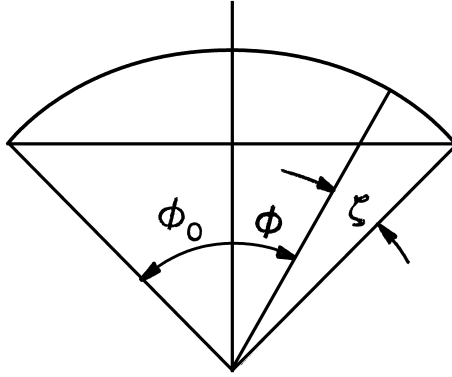


Figure 11-6. Spherical shell angles designation.

The circumferential membrane force is determined from Eq. (11-22) as

$$N_\theta = -\sqrt{2} C_1 e^{-\lambda\zeta} \sin(\lambda\zeta + C_2 - \pi/4). \quad (11-36)$$

The rotation is determined from Eq. (11-28),

$$\alpha = -\frac{2\lambda^2}{Et} C_1 e^{-\lambda\zeta} \cos(\lambda\zeta + C_2). \quad (11-37)$$

The moments are obtained from Eqs. (11-18) and (11-19) using only higher order derivatives

$$\left. \begin{aligned} M_\phi &= -\frac{D}{R} \frac{d\alpha}{d\phi} \\ &= \frac{R}{\sqrt{2}\lambda} C_1 e^{-\lambda\zeta} \sin(\lambda\zeta + C_2 + \pi/4) \end{aligned} \right] \quad (11-38)$$

$$M_\theta = \mu M_\phi. \quad (11-39)$$

The horizontal displacement is obtained from Fig. 11-7 as

$$\begin{aligned} w_h &= v \cos \phi - w \sin \phi \\ &= (v \cot \phi - w) \sin \phi \end{aligned} \quad (11-40)$$

and from Eq. (11-24)

$$w_h = \frac{R}{Et} (N_\theta - \mu N_\phi) \sin \phi \quad (11-41)$$

$$= -\frac{\sqrt{2}\lambda R}{Et} C_1 e^{-\lambda\zeta} \sin(\phi_0 - \zeta) \sin(\lambda\zeta + C_2 - \pi/4). \quad (11-42)$$

Table 11-1 gives various design values for spherical shells due to applied edge loads.

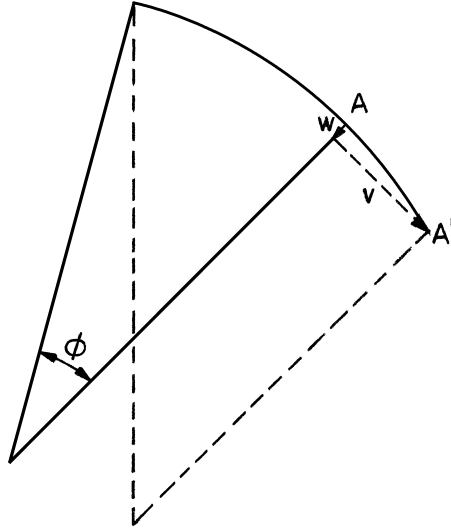


Figure 11-7. Spherical shell displacement.

Example 11-1

Find the location and maximum value of moment M_ϕ due to the horizontal force shown in Fig. 11-8.

Solution

The first boundary condition at the edge is

$$Q = H_0 \sin \phi_0 \quad \text{at} \quad \zeta = 0$$

and from Eq. (11-34)

$$C_1 \sin(-C_2) = H_0 \sin \phi_0. \quad (1)$$

From Eqs. (11-28) and (11-38)

$$M = \frac{-D}{R} \frac{1}{Et} \frac{d^3 Q}{d\zeta^3}.$$

Substituting the third derivative of Eq. (11-34) into this expression gives

$$M_\phi = \frac{-2DC_1\lambda^3 e^{-\lambda\zeta}}{REt} [(\cos C_2)(\cos \lambda\zeta + \sin \lambda\zeta) + (\sin C_2)(\sin \lambda\zeta - \cos \lambda\zeta)].$$

The second boundary condition at the edge is

$$M_0 = 0$$

or,

$$\cos C_2(1) + \sin C_2(-1) = 0. \quad (2)$$

Table 11-1. Edge Loads on Spherical Shells

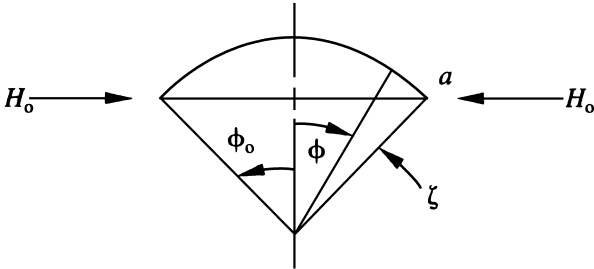
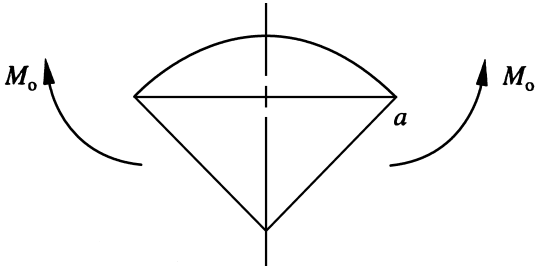
	
Edge Functions $\zeta = 0$ and $\phi = \phi_o$	
α	$-\frac{2H_o\lambda^2}{Et} \sin \phi_o$
w_h	$\frac{H_oR}{Et} \sin \phi_o (2\lambda \sin \phi_o - \mu \cos \phi_o)$
General Functions	
Q	$\sqrt{2}H_o e^{-\lambda\zeta} \sin \phi_o \cos \left(\lambda\zeta + \frac{\pi}{4} \right)$
N_θ	$-2H_o\lambda e^{-\lambda\zeta} \sin \phi_o \cos \lambda\zeta$
M_ϕ	$-\frac{H_oR}{\lambda} e^{-\lambda\zeta} \sin \phi_o \sin \lambda\zeta$
α	$-\frac{H_o}{Et} \left[2\sqrt{2}\lambda^2 e^{-\lambda\zeta} \sin \phi_o \sin \left(\lambda\zeta + \frac{\pi}{4} \right) \right]$
w_h	$\frac{H_oR}{Et} e^{-\lambda\zeta} \sin \phi_o \left[2\lambda \sin \phi_o \cos \lambda\zeta - \sqrt{2}\mu \cos \phi_o \cos \left(\lambda\zeta + \frac{\pi}{4} \right) \right]$
	
Edge Functions $\zeta = 0$ and $\phi = \phi_o$	
α	$\frac{4M_o\lambda^3}{EtR}$
w_h	$-\frac{2M_o\lambda^2}{Et} \sin \phi_o$

Table 11-1. (continued)

General Functions	
Q	$-\frac{2M_o\lambda}{R}e^{-\lambda\zeta}\sin\lambda\zeta$
N_θ	$2\sqrt{2}\frac{M_o\lambda^2}{R}e^{-\lambda\zeta}\cos\left(\lambda\zeta + \frac{\pi}{4}\right)$
M_ϕ	$\sqrt{2}M_o e^{-\lambda\zeta}\sin\left(\lambda\zeta + \frac{\pi}{4}\right)$
α	$\frac{4M_o\lambda^3}{EtR}e^{-\lambda\zeta}\cos\lambda\zeta$
w_h	$-\frac{2M_o\lambda}{Et}e^{-\lambda\zeta}\left[\sqrt{2}\lambda\sin\phi\cos\left(\lambda\zeta + \frac{\pi}{4}\right) + \mu\cos\phi\sin\lambda\zeta\right]$

Notation: $M_\theta = \mu M_\phi$; $N_\phi = -Q \cot \phi$; w_h = horizontal component of deflection; α = rotation; $\zeta = \phi_o - \phi$; $\lambda = \sqrt[4]{(R/t)^2 3(1 - \mu^2)}$; μ = Poisson's ratio. Inward deflection is positive. Positive rotation is in direction of positive moments. Tensile N_ϕ and N_θ are positive. Positive moments cause tension on the inside surface. Inward Q is positive.

From Eq. (2) we get

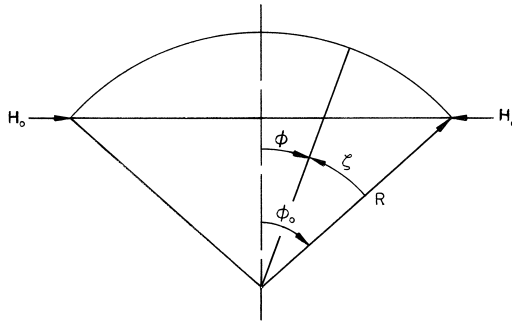
$$C_2 = \pi/4$$

and from Eq. (1) we get

$$C_1 = \frac{2}{\sqrt{2}} H_o \sin \phi_o.$$

The expression for M_ϕ becomes

$$M_\phi = \frac{4DH_o\lambda^3 e^{-\lambda\zeta}}{REt} \sin \phi \sin \lambda\zeta.$$

**Figure 11-8.** Horizontal edge load.

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Substituting

$$D = \frac{Et^3}{12(1 - \mu^2)}$$

into the moment expression results in

$$M_\phi = \frac{H_o R}{\lambda} e^{-\lambda \zeta} \sin \phi_o \sin \lambda \zeta \quad (3)$$

which is the same as that given in Table 11-1.

The location of the maximum moment is obtained from

$$\frac{dM_\phi}{d\zeta} = 0$$

or

$$\zeta = \frac{\pi}{2\lambda}.$$

The maximum moment is obtained from Eq. (3) as

$$M_o = \frac{0.2079 H_o R}{\lambda} \sin \phi_o.$$

Example 11-2

Calculate the required thickness of the hemispherical and cylindrical shells in Fig. 11-9a and determine the discontinuity stress at the junction. Let $p = 200$ psi. Allowable membrane stress is 18 ksi and $\mu = 0.3$.

Solution

The required thickness of the cylindrical shell is

$$t = \frac{pr}{\sigma} = \frac{200 \times 36}{18,000} = 0.40 \text{ inch.}$$

The required thickness of the hemispherical shell is

$$t = \frac{pR}{2\sigma} = \frac{200 \times 36}{2 \times 18,000} = 0.20 \text{ inch.}$$

The discontinuity forces are shown in Fig. 11-9b. The first compatibility equation at point a is

deflection of cylinder due to $p + H_o + M_o$

$$= \text{deflection of hemisphere due to } p + H_o + M_o \quad (1)$$

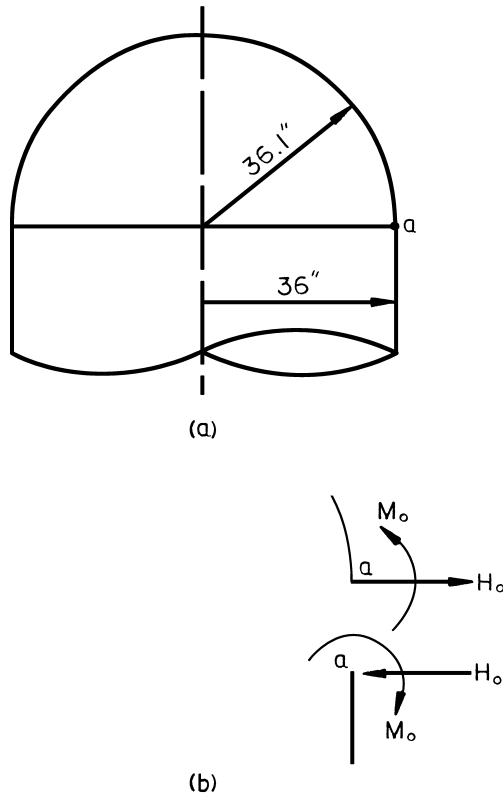


Figure 11-9. Hemispherical-to-cylindrical shell junction.

The expressions for the deflection of the cylinder are obtained from Eq. (10-32) and Table 10-2. For the hemisphere, the expressions are obtained from Eq. (1) of Example 9-3 and Table 11-1. Let outward deflection and clockwise rotation at point a be positive. Hence,

$$\beta = \sqrt[4]{\frac{3(1-\mu^2)}{36^2 \times 0.4^2}} = 0.3387$$

$$D = \frac{E(0.4)^3}{12(1-0.3^2)} = \frac{5.8608E}{1000}$$

$$\lambda = \sqrt[4]{\left(\frac{36.1}{0.2}\right)^2 3(1-0.3^2)} = 17.2695.$$

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Thus Eq. (1) becomes

$$\begin{aligned} & \frac{(200)(36)^2}{E(0.4)}(0.85) - \frac{H_o}{(2)(0.3387)^3(5.8608E/1000)} + \frac{M_o}{(2)(0.3387)^2(5.8608E/1000)} \\ &= \frac{(200)(36.1)^2}{(2)(E)(0.2)}(1 - 0.3) + \frac{H_o(36.1)}{E(0.2)}(2 \times 17.2695) + \frac{2M_o(17.2695)^2}{E(0.2)} \\ 550,800 - 2195.67H_o + 743.67M_o &= 456,123.5 + 6234.29H_o + 2982.36M_o \\ M_o + 3.77H_o &= 42.29. \end{aligned} \quad (2)$$

The second compatibility equation at point a is given by rotation of cylinder due to $p + H_o + M_o$ = rotation of hemisphere due to $p + H_o + M_o$

$$\begin{aligned} 0 - \frac{H_o}{(2)(0.3387)^2(5.8608E/1000)} + \frac{M_o}{(0.3387)(5.8608E/1000)} \\ = 0 - \frac{2H_o(17.2695)^2}{E(0.2)} - \frac{4M_o(17.2695)^3}{E(0.2)(36.1)} \\ 1.50M_o + H_o = 0. \end{aligned} \quad (3)$$

Solving Eqs. (2) and (3) yields

$$M_o = -9.08 \text{ inch-lbs/inch} \quad H_o = 13.63 \text{ lb/inch}$$

The negative sign for the moment indicates that the actual moment is opposite that assumed in Fig. 11-9b.

Cylindrical Shell

$$\begin{aligned} \text{longitudinal bending stress} &= \frac{6M_o}{t^2} = 340 \text{ psi} \\ \text{longitudinal membrane stress} &= \frac{pr}{2t} = 9000 \text{ psi} \\ \text{total longitudinal stress} &= 9340 \text{ psi} \end{aligned}$$

From Eq. (10-13),

$$\begin{aligned} \text{hoop bending stress} &= 0.3 \times 340 = 100 \text{ psi} \\ \text{deflection at point } a &= (550,800 - 2195.67H_o + 743.67M_o)/E \\ &= 514,120/E. \end{aligned}$$

The hoop membrane force is obtained from Eq. (10-9) as

$$N_\theta = \frac{Etw}{r} = 5712 \text{ lbs/inch}$$

$$\text{hoop membrane stress} = \frac{5712}{0.4} = 14,300 \text{ psi}$$

$$\text{total hoop stress} = 14,400 \text{ psi}$$

Hemispherical Shell

$$\begin{aligned} \text{longitudinal stress} &= \frac{pR}{2t} + \frac{6M_o}{t^2} \\ &= \frac{(200)(36.1)}{2(0.2)} + \frac{6(9.08)}{0.2^2} \\ &= 19,410 \text{ psi} \end{aligned}$$

From Table 11-1,

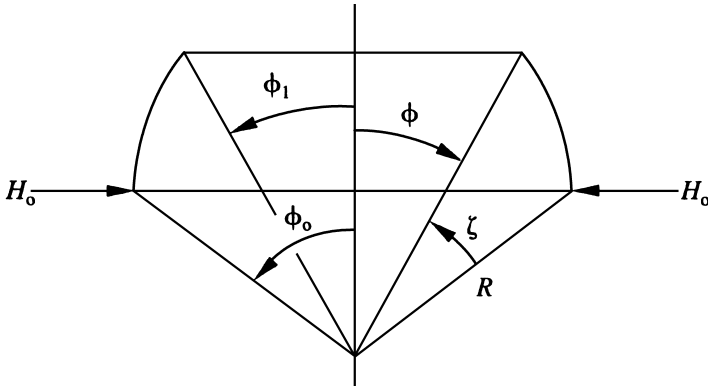
$$\begin{aligned} \text{hoop force} &= \frac{pR}{2} + H_o\lambda - 2M_o\lambda^2/R \\ &= \frac{(200)(36.1)}{2} + (13.63)(17.2695) - \frac{2(9.08)(17.2695)^2}{36.1} \\ &= 18,480 \text{ psi} \end{aligned}$$

If a spherical shell has an axisymmetric hole, and is subjected to edge loads, then Eq. (11-33) must be used to determine the constants K_1 through K_4 . Other design functions are then established from the various equations derived. Table 11-2 lists various equations for open spherical shells.

Problems

- 11-1 Derive Eq. (11-35)
- 11-2 What is the maximum stress in the spherical shell and flat plate due to internal pressure of 600 psi? Let $E = 16,000$ ksi and $\mu = 0.0$.
- 11-3 Determine the discontinuity stress at the spherical-to-cylindrical junction shown. The dimensions of the stiffening ring at the junction are 4 inches $\times$ 3/4 inch. Let $E = 30,000$ ksi and $\mu = 0.30$.
- 11-4 The heating compartment between the two spherical shells is subjected to 100 psi pressure. Find the discontinuity stress in the top and bottom spherical shells. $E = 27,000$ ksi and $\mu = 0.28$.
- 11-5 Determine the length $L = \zeta R$ where the moment diminishes to 1% of moment M_o applied at the free edge. How does this length compare with that in Example 8-1 for cylindrical shells?
- 11-6 Derive Eq. (11-42).

Table 11-2. Edge loads in open spherical shells (Baker et al. 1968)



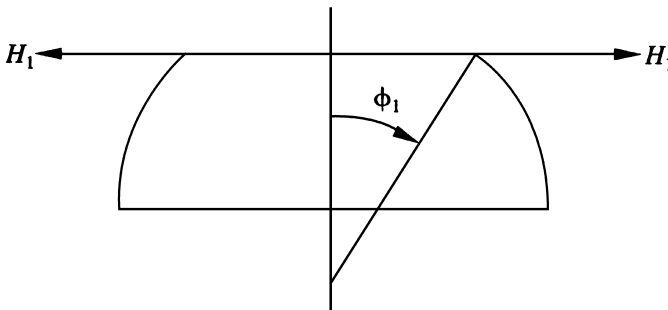
$$Q = H_o \sin \phi_o \left[-K_7(\zeta) - \frac{K_4}{K_1} K_{10}(\zeta) + \frac{K_2}{K_1} K_8(\zeta) \right]$$

$$N_\theta = -H_o \lambda \sin \phi_o \left[-K_9(\zeta) - \frac{2K_4}{K_1} K_7(\zeta) + \frac{K_5}{K_1} K_{10}(\zeta) \right]$$

$$M_\phi = -H_o \frac{R}{2\lambda} \sin \phi_o \left[-K_{10}(\zeta) + \frac{2K_4}{K_1} K_8(\zeta) - \frac{K_2}{K_1} K_9(\zeta) \right]$$

$$\alpha = -H_o \frac{2\lambda^2}{Et} \sin \phi_o \left[-K_8(\zeta) - \frac{K_4}{K_1} K_9(\zeta) + \frac{K_2}{K_1} K_7(\zeta) \right]$$

$$w_h = H_o \frac{R\lambda}{Et} \sin \phi \sin \phi_o \left[-K_9(\zeta) - \frac{2K_4}{K_1} K_7(\zeta) + \frac{K_2}{K_1} K_{10}(\zeta) \right]$$

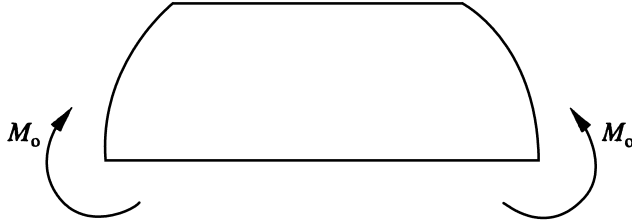


$$Q = -H_1 \sin \phi_1 \left[-\frac{K_9}{K_1} K_{10}(\zeta) - \frac{2K_8}{K_1} K_8(\zeta) \right]$$

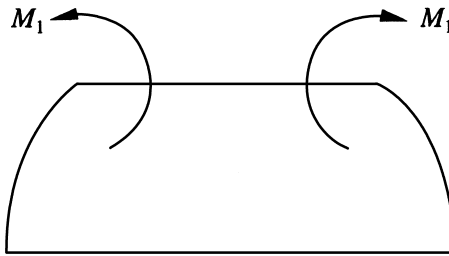
$$N_\theta = H_1 2\lambda \sin \phi_1 \left[-\frac{K_9}{K_1} K_7(\zeta) + \frac{K_8}{K_1} K_{10}(\zeta) \right]$$

Table 11-2. (continued)

$$\begin{aligned}
 M_\phi &= H_1 \frac{R}{\lambda} \sin \phi_1 \left[\frac{K_9}{K_1} K_8(\zeta) - \frac{K_8}{K_1} K_9(\zeta) \right] \\
 \alpha &= H_1 \frac{2\lambda^2}{Et} \sin \phi_1 \left[\frac{K_9}{K_1} K_9(\zeta) + \frac{2K_8}{K_1} K_7(\zeta) \right] \\
 w_h &= -H_1 \frac{2R\lambda}{Et} \sin \phi \sin \phi_1 \left[\frac{K_9}{K_1} K_7(\zeta) - \frac{K_8}{K_1} K_{10}(\zeta) \right]
 \end{aligned}$$



$$\begin{aligned}
 Q &= -M_o \frac{2\lambda}{R} \left[\frac{K_6}{K_1} K_{13}(\zeta) + \frac{K_5}{K_1} K_{14}(\zeta) - \frac{K_3}{K_1} K_8(\zeta) \right] \\
 N_\theta &= M_o \frac{2\lambda^2}{R} \left[\frac{K_6}{K_1} K_{12}(\zeta) + \frac{K_5}{K_1} K_{11}(\zeta) - \frac{K_3}{K_1} K_{10}(\zeta) \right] \\
 M_\phi &= M_o \left[\frac{K_6}{K_1} K_{11}(\zeta) - \frac{K_5}{K_1} K_{12}(\zeta) + \frac{K_3}{K_1} K_9(\zeta) \right] \\
 \alpha &= M_o \frac{4\lambda^3}{EtR} \left[\frac{K_6}{K_1} K_{14}(\zeta) - \frac{K_5}{K_1} K_{13}(\zeta) - \frac{K_3}{K_1} K_7(\zeta) \right] \\
 w_h &= -M_o \frac{2\lambda^2}{Et} \sin \phi \left[\frac{K_6}{K_1} K_{12}(\zeta) + \frac{K_5}{K_1} K_{11}(\zeta) - \frac{K_3}{K_1} K_{10}(\zeta) \right]
 \end{aligned}$$



$$\begin{aligned}
 Q &= -M_1 \frac{2\lambda}{R} \left[-\frac{K_8}{K_1} K_{10}(\zeta) + \frac{K_{10}}{K_1} K_8(\zeta) \right] \\
 N_\theta &= M_1 \frac{2\lambda^2}{R} \left[-\frac{2K_8}{K_1} K_7(\zeta) + \frac{K_{10}}{K_1} K_{10}(\zeta) \right]
 \end{aligned}$$

Table 11-2. (continued)

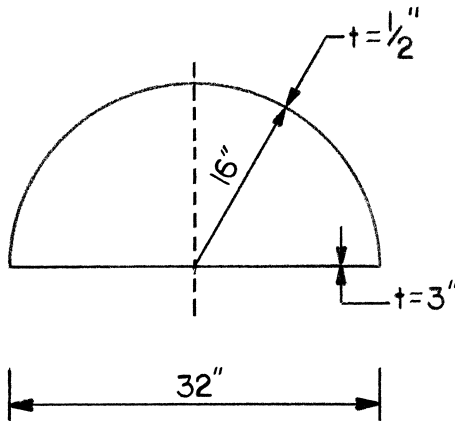
$$\begin{aligned}
 M_\phi &= M_1 \left[\frac{2K_8}{K_1} K_8(\zeta) - \frac{K_{10}}{K_1} K_9(\zeta) \right] \\
 \alpha &= M_1 \frac{4\lambda^3}{EtR} \left[\frac{K_8}{K_1} K_9(\zeta) + \frac{K_{10}}{K_1} K_7(\zeta) \right] \\
 w_h &= -M_1 \frac{2\lambda^2}{Et} \sin \phi \left[-\frac{2K_8}{K_1} K_7(\zeta) + \frac{K_{10}}{K_1} K_{10}(\zeta) \right]
 \end{aligned}$$

Notation: $M_\theta = \mu M_\phi$; $N_\phi = -Q \cot \phi$; w_h = horizontal component of deflection; α = rotation; $K_1 = \sinh^2 \lambda - \sin^2 \lambda$; $K_2 = \sinh^2 \lambda + \sin^2 \lambda$; $K_3 = \sinh \lambda \cosh \lambda + \sin \lambda \cos \lambda$; $K_4 = \sinh \lambda \cosh \lambda - \sin \lambda \cos \lambda$; $K_5 = \sin^2 \lambda$; $K_6 = \sinh^2 \lambda$; $K_7 = \cosh \lambda \cos \lambda$; $K_8 = \sinh \lambda \sin \lambda$; $K_8(\zeta) = \sinh \zeta \lambda \sin \zeta \lambda$; $K_9 = \cosh \lambda \sin \lambda - \sinh \lambda \cos \lambda$; $K_9(\zeta) = \cosh \zeta \lambda \sin \zeta \lambda - \sinh \zeta \lambda \cos \zeta \lambda$; $K_{10} = \cosh \lambda \sin \lambda + \sinh \lambda \cos \lambda$; $K_{10}(\zeta) = \cosh \zeta \lambda \sin \zeta \lambda + \sinh \zeta \lambda \cos \zeta \lambda$; $K_{11}(\zeta \lambda) = \cosh \zeta \lambda \cos \zeta \lambda - \sinh \zeta \lambda \sin \zeta \lambda$; $K_{12} = \cosh \lambda \cos \lambda + \sinh \lambda \sin \lambda$; $K_{12}(\zeta \lambda) = \cosh \zeta \lambda \cos \zeta \lambda + \sinh \zeta \lambda \sin \zeta \lambda$; $K_{13}(\zeta \lambda) = \cosh \zeta \lambda \sin \zeta \lambda$; $K_{14}(\zeta \lambda) = \sinh \zeta \lambda \cos \zeta \lambda$.

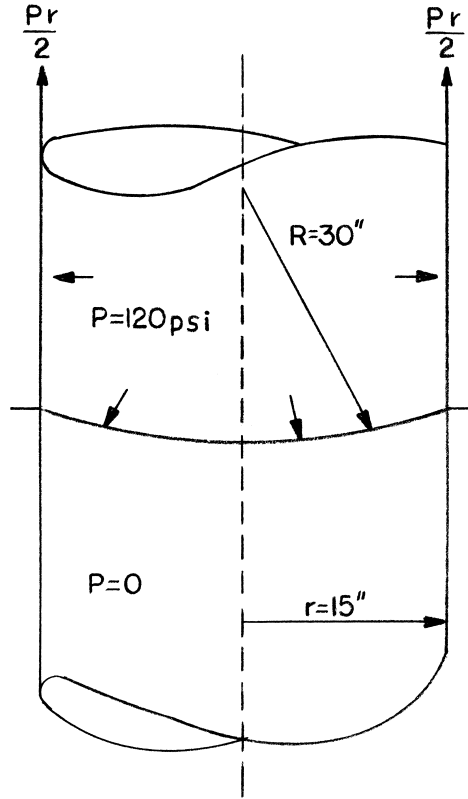
11-3 Conical Shells

The derivation of the expressions for the bending moments in conical shells is obtained from the general equations of Section 11-1. In this case the angle is constant as shown in Fig. 11-10. Equations (11-5), (11-6), (11-7), (11-12), (11-13), (11-14), (11-20), and (11-21) have to be rewritten for conical shells with the following substitutions

$$\begin{aligned}
 \phi &= \frac{\pi}{2} - \alpha, \quad r_1 = \infty, \quad r_2 = s \tan \alpha \\
 ds &= r_1 d\phi.
 \end{aligned}$$



Prob. 11-2. Hemispherical shell connected to a circular plate.



Prob. 11-3. Interior partition in a cylindrical shell.

The solutions of the resulting eight equations (Flügge 1967) involve Bessel functions. These solutions are too cumbersome to use on a regular basis. However, simplified asymptotic solutions, similar to those developed for spherical shells, can be developed for the large end of conical shells with $\beta > 11$ where

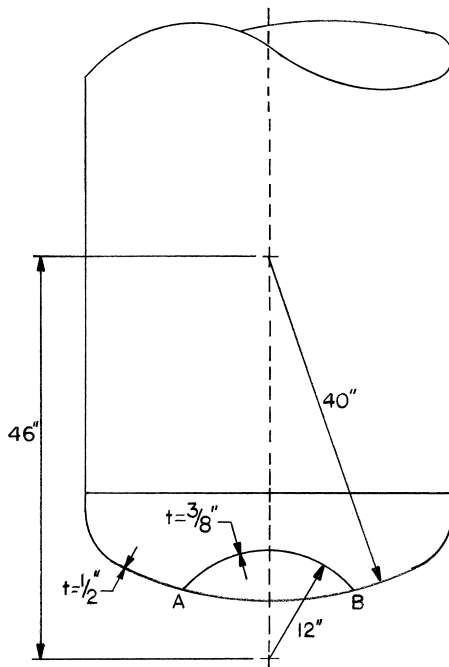
$$\beta = 2\sqrt[4]{3(1-\mu^2)}\sqrt{\frac{s \cot \alpha}{t}}. \quad (11-43)$$

This range of β is common for most conical shells encountered in industry.

$$Q_s = \frac{e^\beta}{2^{3/4}s\sqrt{\pi\beta}} \left[C_1 \cos \left(\beta - \frac{\pi}{8} \right) + C_2 \sin \left(\beta - \frac{\pi}{8} \right) \right] \quad (11-44)$$

$$M_s = \frac{e^\beta}{\sqrt{2}\beta\sqrt{\pi\beta}} \left[C_1 \sin \left(\beta + \frac{\pi}{8} \right) - C_2 \cos \left(\beta + \frac{\pi}{8} \right) \right] \quad (11-45)$$

$$M_\theta = \mu M_s \quad (11-46)$$



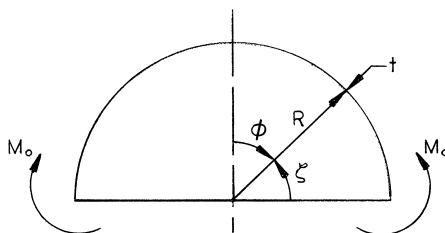
Prob. 11-4. Internal compartment.

$$N_{\theta} = \frac{-\sqrt{\beta} e^{\beta} \tan \alpha}{2 \sqrt[4]{2} \sqrt{\pi s}} \left[C_1 \cos \left(\beta + \frac{\pi}{8} \right) + C_2 \sin \left(\beta + \frac{\pi}{8} \right) \right] \quad (11-47)$$

$$N_s = Q_s \tan \alpha \quad (11-48)$$

$$w = \frac{s \sin \alpha}{Et} (N_{\theta} - \mu N_s) \quad (11-49)$$

where C_1 and C_2 are obtained from the boundary conditions.



Prob. 11-5. Moment applied at free edge.

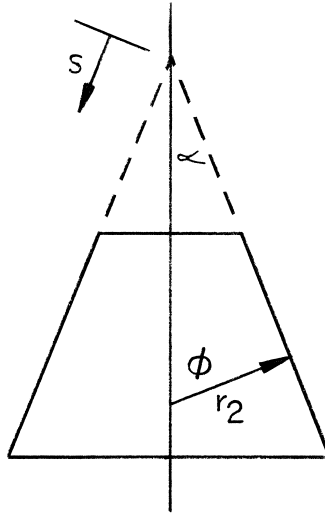


Figure 11-10. Conical shell.

Application of Eqs. (11-43) to (11-49) to edge forces of a full cone is given in Table 11-3. Table 11-4 lists various equations for truncated cones.

Example 11-3

Find the discontinuity forces in the cone Fig. 11-11a, due to an internal pressure of 60 psi. Let $E = 30,000$ ksi and $\mu = 0.30$.

Solution

The deflection w at point A due to internal pressure, Fig. 11-11b, is obtained from Eq. (10-32) by using the radius $r/\cos \alpha$ rather than r . Hence, radial deflection is expressed as

$$w = \frac{-pr^2}{Et \cos^2 \alpha} (1 - \mu/2). \quad (1)$$

The horizontal deflection is

$$\begin{aligned} w_H &= \frac{-pr^2}{Et \cos \alpha} (1 - \mu/2) \\ &= \frac{-(60)(54)^2(1 - 0.3/2)}{30,000,000(0.25)(0.8)} \end{aligned}$$

$$w_H = -0.0248 \text{ inch.}$$

Table 11-3. Edge loads in conical shells (Jawad and Farr 1989)

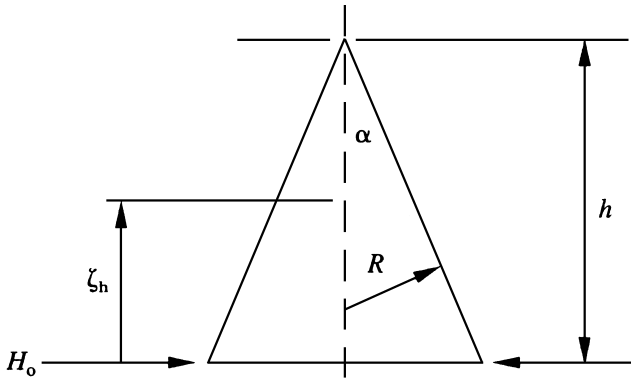
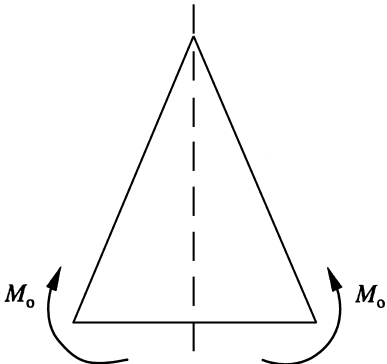
	
Edge Functions $\zeta = 0$	
θ	$-\frac{H_o h^2}{2D\beta^2 \cos \alpha}$
w_h	$-\frac{H_o h^3}{2D\beta^3 \cos \alpha} \left[1 - \frac{\mu h \tan \alpha}{2R\beta \cos \alpha} \right]$
General Functions	
Q	$\sqrt{2}H_o e^{-\beta\zeta} \cos \alpha \cos \left(\beta\zeta + \frac{\pi}{4} \right)$
N_θ	$-\frac{2H_o R^2 \beta \cos^2 \alpha}{h} e^{-\beta\zeta} \cos \beta\zeta$
M_ϕ	$-\frac{H_o h}{\beta} e^{-\beta\zeta} \sin \beta\zeta$
θ	$-\frac{H_o h^2}{\sqrt{2}D\beta^2 \cos \alpha} e^{-\beta\zeta} \sin \left(\beta\zeta + \frac{\pi}{4} \right)$
w_h	$\frac{H_o h^3 e^{-\beta\zeta}}{2D\beta^3 \cos \alpha} \left[\cos \beta\zeta - \frac{\mu h}{2R\beta} \tan \alpha \cos \left(\beta\zeta + \frac{\pi}{4} \right) \right]$
	

Table 11-3. (continued)

Edge Functions $\zeta = 0$	
θ	$\frac{M_0 h}{D\beta \cos \alpha}$
w_h	$-\frac{M_0 h^2}{2D\beta^2 \cos \alpha}$
General Functions	
Q	$-\frac{2M_0\beta}{h} \cos \alpha e^{-\beta\zeta} \sin \beta\zeta$
N_θ	$-\frac{4R\beta^2 M_0}{\sqrt{2}h^2} \cos^2 \alpha e^{-\beta\zeta} \cos\left(\beta\zeta + \frac{\pi}{4}\right)$
M_ϕ	$\sqrt{2}M_0 e^{-\beta\zeta} \sin\left(\beta\zeta + \frac{\pi}{4}\right)$
θ	$\frac{M_0 h}{D\beta \cos \alpha} e^{-\beta\zeta} \cos \beta\zeta$
w_h	$-\frac{M_0 h^2}{2D\beta^2 \cos \alpha} e^{-\beta\zeta} \left[\sqrt{2} \cos\left(\beta\zeta + \frac{\pi}{4}\right) + \frac{\mu h}{\beta R \cos \alpha} \tan \alpha \sin \beta\zeta \right]$

Notation: $M_\theta = \mu M_\phi$; $N_\phi = -Q \cot \alpha$; w_h = horizontal component of deflection; θ = rotation; $\beta = (h/(\sqrt{Rt} \sin \alpha)) \sqrt[4]{3(1 - \mu^2)}$; μ = Poisson's ratio. Inward deflection is positive. Positive rotation is in direction of positive moments. Tensile N_ϕ and N_θ are positive. Positive moments cause tension on the inside surface. Inward Q is positive.

Table 11-4. Edge loads in open conical shells (Baker et al. 1968)

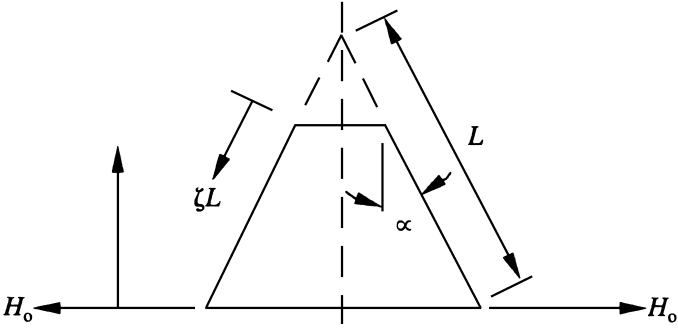
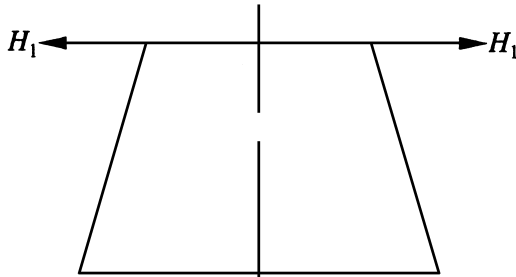
	
Q	$-H_o \cos \alpha \left[-\frac{K_9}{K_1} K_{10}(\zeta) + \frac{2K_8}{K_1} K_8(\zeta) \right]$
N_θ	$2H_o x_m \sin \alpha \left[-\frac{K_9}{K_1} K_7(\zeta) + \frac{K_8}{K_1} K_{10}(\zeta) \right]$

Table 11-4. (continued)

M_s	$H_0 \frac{1}{\beta} \cos \alpha \left[\frac{K_9}{K_1} K_8(\zeta) - \frac{K_8}{K_1} K_9(\zeta) \right]$
θ	$H_0 \frac{1}{2D\beta^2} \cos \alpha \left[-\frac{K_9}{K_1} K_9(\zeta) + \frac{2K_8}{K_1} K_7(\zeta) \right]$
w_h	$-H_0 \frac{1}{2D\beta^3} \cos^2 \alpha \left[-\frac{K_9}{K_1} K_7(\zeta) + \frac{K_8}{K_1} K_{10}(\zeta) \right]$



Q	$-H_1 \cos \alpha \left[K_7(\zeta) - \frac{K_4}{K_1} K_{10}(\zeta) + \frac{K_2}{K_1} K_8(\zeta) \right]$
N_θ	$H_1 x_m \beta \sin \alpha \left[K_9(\zeta) + \frac{2K_4}{K_1} K_7(\zeta) - \frac{K_2}{K_1} K_{10}(\zeta) \right]$
M_s	$H_1 \frac{1}{2} \cos \alpha \left[K_{10}(\zeta) - \frac{2K_4}{K_1} K_8(\zeta) + \frac{K_2}{K_1} K_9(\zeta) \right]$
θ	$H_1 \frac{\cos \alpha}{2D\beta^2} \left[-K_8(\zeta) + \frac{K_4}{K_1} K_9(\zeta) + \frac{K_2}{K_1} K_7(\zeta) \right]$
w_h	$-H_1 \frac{\cos^2 \alpha}{4D\beta^3} \left[K_9(\zeta) + \frac{2K_4}{K_1} K_7(\zeta) - \frac{K_2}{K_1} K_{10}(\zeta) \right]$

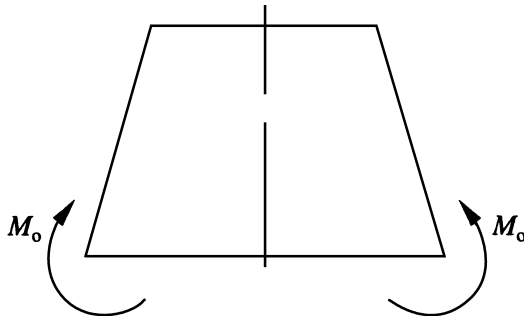
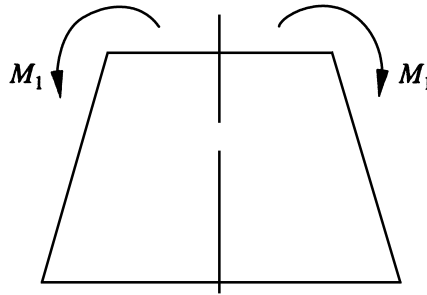


Table 11-4. (continued)

Q	$-M_0 2\beta \left[\frac{K_8}{K_1} K_{10}(\zeta) - \frac{K_{10}}{K_1} K_8(\zeta) \right]$
N_θ	$M_0 2\beta^2 x_m \tan \alpha \left[\frac{2K_8}{K_1} K_7(\zeta) - \frac{K_{10}}{K_1} K_{10}(\zeta) \right]$
M_s	$M_0 \left[\frac{2K_8}{K_1} K_8(\zeta) - \frac{K_{10}}{K_1} K_9(\zeta) \right]$
θ	$M_0 \frac{1}{D\beta} \left[\frac{K_8}{K_1} K_9(\zeta) + \frac{K_{10}}{K_1} K_7(\zeta) \right]$
w_h	$-M_0 \frac{\cos \alpha}{2D\beta^2} \left[\frac{2K_8}{K_1} K_7(\zeta) - \frac{K_{10}}{K_1} K_{10}(\zeta) \right]$



Q	$-M_1 2\beta \left[\frac{K_6}{K_1} K_{13}(\zeta) + \frac{K_5}{K_1} K_{14}(\zeta) - \frac{K_3}{K_1} K_8(\zeta) \right]$
N_θ	$M_1 2\beta^2 x_m \tan \alpha \left[\frac{K_6}{K_1} K_{12}(\zeta) + \frac{K_5}{K_1} K_{11}(\zeta) - \frac{K_3}{K_1} K_{10}(\zeta) \right]$
M_s	$M_1 \left[\frac{K_6}{K_1} K_{11}(\zeta) - \frac{K_5}{K_1} K_{12}(\zeta) + \frac{K_3}{K_1} K_9(\zeta) \right]$
θ	$M_1 \frac{1}{D\beta} \left[\frac{K_6}{K_1} K_{14}(\zeta) - \frac{K_5}{K_1} K_{13}(\zeta) - \frac{K_3}{K_1} K_7(\zeta) \right]$
w_h	$-M_1 \frac{\cos \alpha}{2D\beta^2} \left[\frac{K_6}{K_1} K_{12}(\zeta) + \frac{K_5}{K_1} K_{11}(\zeta) - \frac{K_3}{K_1} K_{10}(\zeta) \right]$

Notation: $M_0 = \mu M_\phi$; $N_\theta = -Q \cot \alpha$; w_h = horizontal component of deflection; x_m = Slanted length measured from the apex to mid-distance of the truncated cone; θ = rotation; $\beta = \sqrt[4]{3(1-\mu^2)}/\sqrt{(x_m \tan \alpha)}$; $K_1 = \sinh^2 \beta L - \sin^2 \beta L$; $K_2 = \sinh^2 \beta L + \sin^2 \beta L$; $K_3 = \sinh \beta L \cosh \beta L + \sin \beta L \cos \beta L$; $K_4 = \sinh \beta L \cosh \beta L - \sin \beta L \cos \beta L$; $K_5 = \sin^2 \beta L$; $K_6 = \sinh^2 \beta L$; $K_7 = \cosh \beta L \cos \beta L$; $K_7(\zeta) = \cosh \beta L \zeta \cos \beta L \zeta$; $K_8 = \sinh \beta L \sin \beta L$; $K_8(\zeta) = \sinh \beta L \zeta \sin \beta L \zeta$; $K_9 = \cosh \beta L \sin \beta L - \sinh \beta L \cos \beta L$; $K_9(\zeta) = \cosh \beta L \zeta \sin \beta L \zeta - \sinh \beta L \zeta \cos \beta L \zeta$; $K_{10} = \cosh \beta L \sin \beta L + \sinh \beta L \cos \beta L$; $K_{10}(\zeta) = \cosh \beta L \zeta \sin \beta L \zeta + \sinh \beta L \zeta \cos \beta L \zeta$; $K_{11}(\zeta) = \cosh \beta L \zeta \cos \beta L \zeta - \sinh \beta L \zeta \sin \beta L \zeta$; $K_{12}(\zeta) = \cosh \beta L \zeta \cos \beta L \zeta + \sinh \beta L \zeta \sin \beta L \zeta$; $K_{13}(\zeta) = \cosh \beta L \zeta \sin \beta L \zeta$; $K_{14}(\zeta) = \sinh \beta L \zeta \cos \beta L \zeta$.

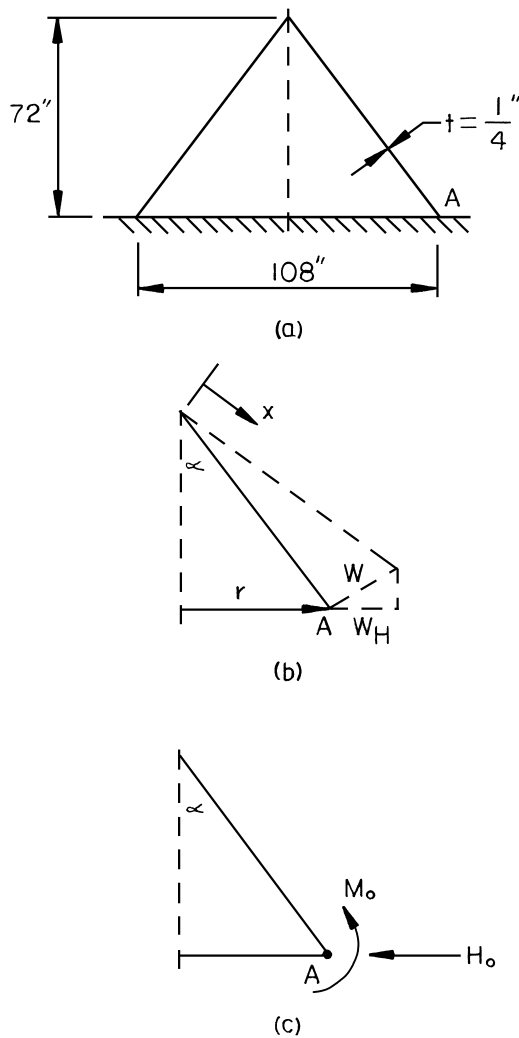


Figure 11-11. Cone with fixed edge.

The rotation, θ , at point A is obtained by writing Eq. (1) as

$$w = \frac{-px^2 \sin^2 \alpha}{Et \cos^2 \alpha} (1 - \mu/2)$$

and from Eq. (11-12) with $v = 0$, and $dx = r_1 d\phi$

$$\theta = -\frac{dw}{r_1 d\phi}$$

$$\begin{aligned}
 &= -\frac{dw}{dx} \\
 \theta &= \frac{2px \sin^2 \alpha}{Et \cos^2 \alpha} (1 - \mu/2) \\
 &= \frac{2pr \sin \alpha}{Et \cos^2 \alpha} (1 - \mu/2) \\
 &= \frac{2(60)(54)(0.6)(0.85)}{30,000,000(0.25)(0.64)} \\
 \theta &= 0.000689 \text{ radians.}
 \end{aligned}$$

From Table 11-3,

$$\begin{aligned}
 R &= r/\cos \alpha = 54/0.8 \\
 &= 67.50
 \end{aligned}$$

$$\begin{aligned}
 \beta &= \frac{72}{\sqrt{67.50 \times 0.25}} \sqrt[4]{3(1 - 0.3^2)} \\
 &= 37.549
 \end{aligned}$$

$$D = 42,926.$$

From Fig. 11-11c,

$$\text{deflection due to } p + M_o + H_o = 0$$

or,

$$\begin{aligned}
 -0.0248 - \frac{hM_o}{2D\beta^2 \cos \alpha} + \frac{h^3 H_o}{2D\beta^3 \cos \alpha} \left(1 - \frac{\mu h \tan \alpha}{2R\beta \cos \alpha} \right) &= 0 \\
 M_o - 1.910H_o &= -463.21.
 \end{aligned} \tag{2}$$

Similarly,

$$\text{rotation due to } p + M_o + H_o = 0$$

$$-0.000689 - \frac{hM_o}{D\beta \cos \alpha} + \frac{h^2 H_o}{2D\beta^2 \cos \alpha} = 0$$

or

$$M_o - 0.959H_o = -12.339. \tag{3}$$

Solving Eqs. (2) and (3) results in

$$M_o = 442.3 \text{ inch-lb/inch.}$$

$$H_o = 474.1 \text{ lb/inch.}$$

Problems

11-7 Find the discontinuity forces at points A , B , and C due to 500 psi pressure in the cone compartment ABC . Let $E = 25,000$ ksi and $\mu = 0.31$ in the cylinder and flat plate. Let $E = 30,000$ ksi and $\mu = 0.29$ in the cone.

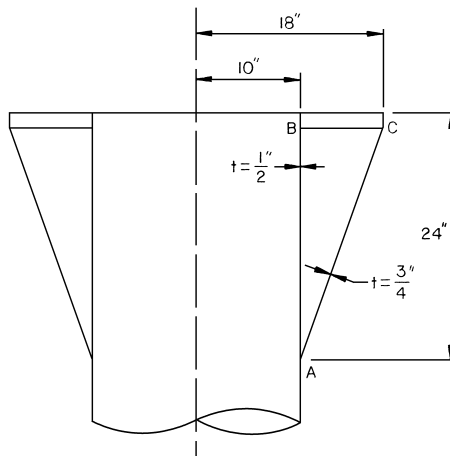
11-8 Find the discontinuity forces at point A due to a 100 psi pressure in compartment AOB . Assume the 40-inch head to be large enough so that the discontinuity forces at A are insignificant at the knuckle region. Let $E = 16,000$ ksi and $\mu = 0.30$.

11-9 Find the discontinuity forces at the cone to flat plate junction. Let $p = 60$ psi, $E = 20,000$ ksi, $\mu = 0.25$ and $\alpha = 36.87^\circ$.

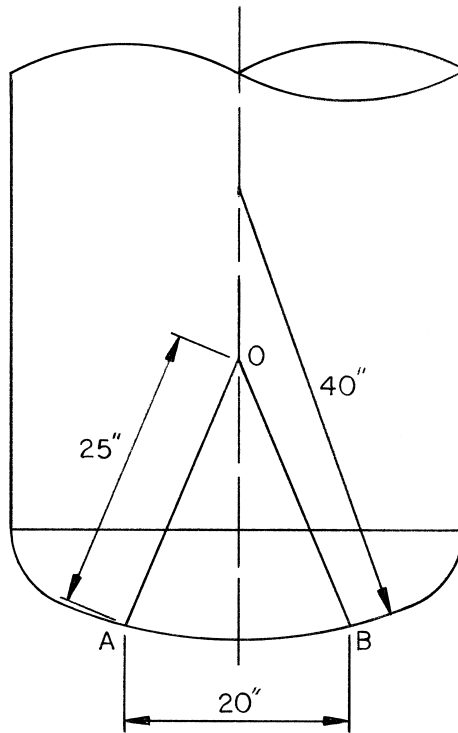
11-4 Design Considerations

Numerous references are available for determining the bending stresses in spherical and conical shells due to various edge conditions. One of the most excessive coverages is given by Baker (Baker et al. 1968). Flugge (Flugge 1967) has extensive tables for solving the differential equations for spherical and conical shells.

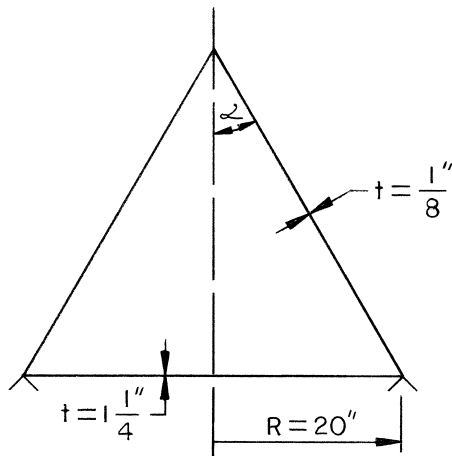
When the configuration of the shell is other than spherical or conical, the classical methods discussed in this chapter become impractical to use and other more general methods such as finite element analysis are employed. It has also been shown by Baker that for some configurations such as those shown in



Prob. 11-7. Cylindrical shell with conical jacket.



Prob. 11-8. Pressurized conical compartment.



Prob. 11-9. Conical shell attached to a circular plate.

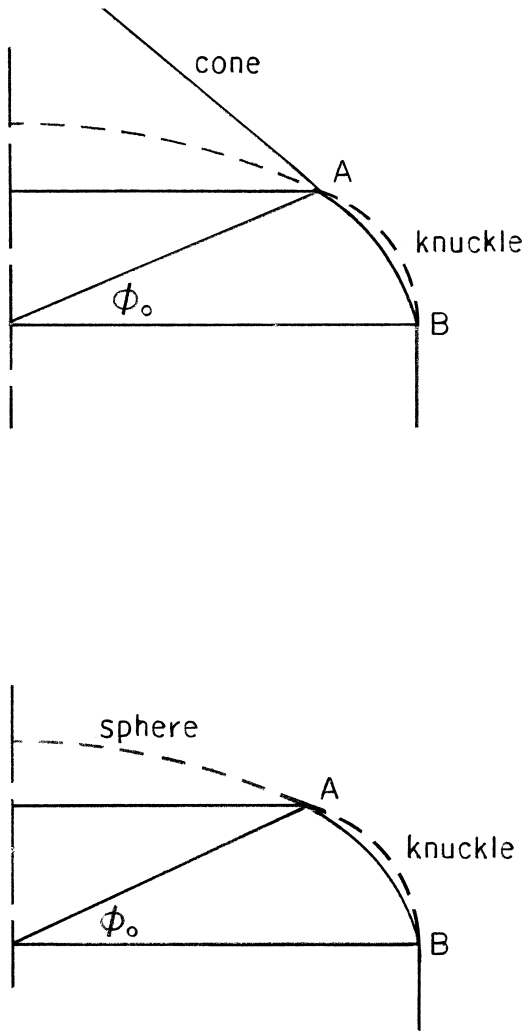


Figure 11-12. Geometric approximations.

Fig. 11-12, the discontinuity forces in the region between points *A* and *B* can be approximated by using the spherical equations derived here for the knuckle portion as long as the angle ϕ_o is larger than about 20 degrees.

12

Various Structures

12-1 Introduction

Many shell structures, Fig. 12-1, have shapes other than cylindrical or spherical. These include hyperbolic paraboloids, Fig. 12-2a, elliptic paraboloids, Fig. 12-2b, barrel structures, Fig. 12-3a, and folded plates, Fig. 12-3b. These structures are characterized by having a small depth to length ratio. They are referred to as shallow shells and the derivation of their pertinent equations is based on the following assumptions

1. The depth to length ratio is less than about 0.20.
2. Forces are mainly due to uniformly distributed dead and live loads.
3. Forces are projected on a rectangular coordinate system for ease of calculations.
4. Thickness of the shell is small compared to depth or length.

In this chapter a brief discussion is presented for the analysis of each of the roof shapes mentioned above. A more complete treatment of the design and detail of these roof structures is discussed in the references cited in this chapter.

We begin the derivation of the equations needed in the analysis of hyperbolic and elliptic paraboloids by expressing the membrane forces in shells of double curvature, Fig. 8-4, in terms of rectangular coordinates, Fig. 12-4, rather than polar coordinates. Also, it will be assumed that the only significant forces acting on roof shells are those due to dead and live loads. These loads are assumed to act in the z -direction only for shallow shells. Projecting the forces of the infinitesimal element $ABCD$ in the plane $EFGH$, Fig. 12-4, and summing forces in the x -, y -, and z -axes, respectively, gives

$$\frac{\partial N_x}{\partial x} dx \frac{dy}{\cos \alpha} \cos \phi + \frac{\partial N_{yx}}{\partial y} dy \frac{dx}{\cos \phi} \cos \phi = 0 \quad (12-1)$$

$$\frac{\partial N_y}{\partial y} dy \frac{dx}{\cos \phi} \cos \alpha + \frac{\partial N_{xy}}{\partial x} dx \frac{dy}{\cos \alpha} \cos \alpha = 0 \quad (12-2)$$



Figure 12-1. Church of the Priority, St. Louis, MO. (Courtesy of Anthony Coombs.)

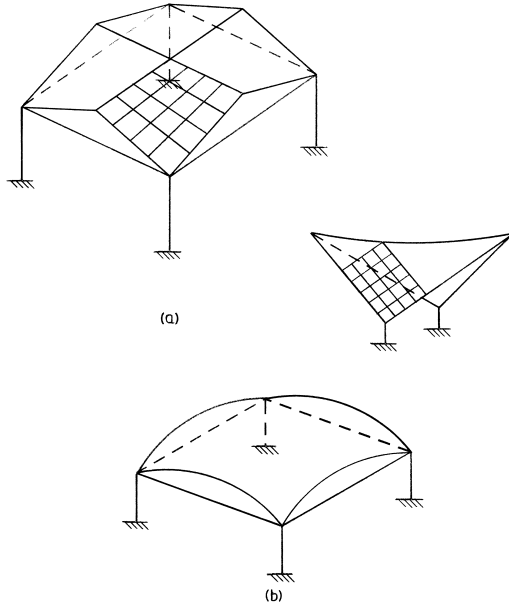


Figure 12-2. Various paraboloid roofs.

and

$$\begin{aligned} \frac{\partial N_x}{\partial x} dx \frac{dy}{\cos \phi} \sin \phi + \frac{\partial N_{xy}}{\partial x} dx \frac{dy}{\cos \alpha} \sin \alpha + \frac{\partial N_y}{\partial y} dy \frac{dx}{\cos \phi} \sin \theta \\ + \frac{\partial N_{xy}}{\partial y} dy \frac{dx}{\cos \phi} \sin \phi - p_z \frac{dx}{\cos \phi} \frac{dy}{\cos \alpha} \sin \beta = 0. \end{aligned} \quad (12-3)$$

If we define

$$\left. \begin{aligned} \bar{N}_x &= N_x \frac{\cos \phi}{\cos \alpha} \\ \bar{N}_y &= N_y \frac{\cos \alpha}{\cos \phi} \\ \bar{p}_z &= p_z \frac{\sin \beta}{\cos \phi \cos \alpha} \end{aligned} \right] \quad (12-4)$$

and substitute these three expressions into Eqs. (12-1) through (12-3) we get

$$\frac{\partial \bar{N}_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (12-5)$$

$$\frac{\partial \bar{N}_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \quad (12-6)$$

$$\frac{\partial \bar{N}_x}{\partial x} \tan \phi + \frac{\partial \bar{N}_y}{\partial y} \tan \alpha + \frac{\partial N_{xy}}{\partial x} \tan \alpha + \frac{\partial N_{xy}}{\partial y} \tan \phi = \bar{p}_z. \quad (12-7)$$

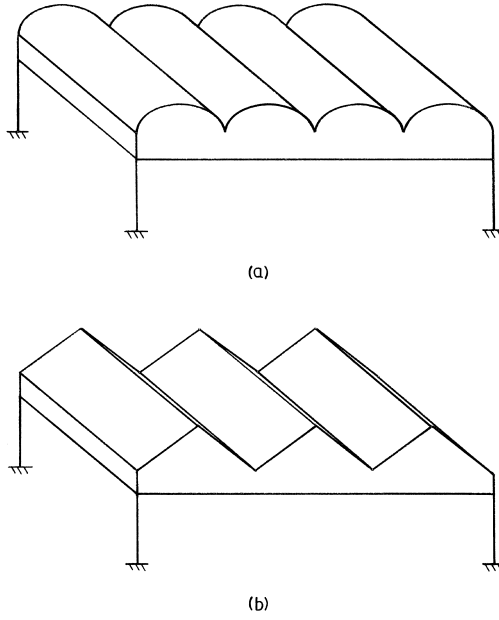


Figure 12-3. Barrel and folded roofs.

Equation (12-7) can further be simplified by substituting the quantities $\tan \phi = \partial z / \partial x$, $\tan \alpha = \partial z / \partial y$, and Eqs. (12-5) and (12-6) into it. This yields

$$\bar{N}_x \frac{\partial^2 z}{\partial x^2} + 2N_{xy} \frac{\partial^2 z}{\partial x \partial y} + \bar{N}_y \frac{\partial^2 z}{\partial y^2} = \bar{p}_z. \quad (12-8)$$

Equations (12-5), (12-6), and (12-8) are the three basic equations for analyzing structures such as hyperbolic paraboloids and elliptic paraboloids. Their solution is obtained by defining a stress function, F , of the form

$$N_{xy} = - \frac{\partial^2 F}{\partial x \partial y}. \quad (12-9)$$

Substituting this expression into Eqs. (12-5), (12-6), and (12-8) results in

$$\bar{N}_x = \frac{\partial^2 F}{\partial y^2} \quad (12-10)$$

$$\bar{N}_y = \frac{\partial^2 F}{\partial x^2} \quad (12-11)$$

$$\frac{\partial^2 F}{\partial y^2} \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 z}{\partial y^2} = \bar{p}_z. \quad (12-12)$$

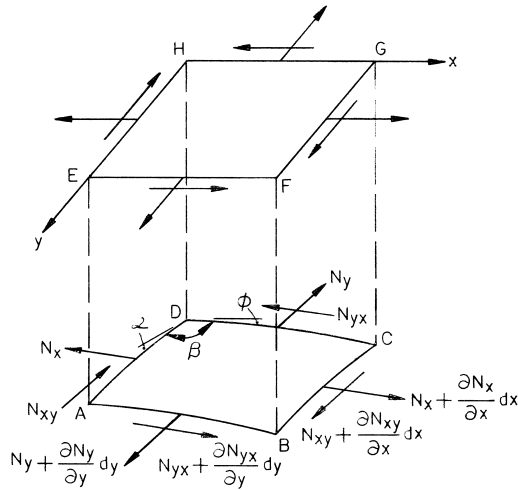


Figure 12-4. Membrane forces.

The solution of Eqs. (12-10) through (12-12) depends on the geometry of the specific shell being analyzed.

Problems

- 12-1 Derive Eqs. (12-1), (12-2), and (12-3).
- 12-2 Derive Eqs. (12-5), (12-6), and (12-7).
- 12-3 Derive Eq. (12-8).

12-2 Hyperbolic Paraboloid Shells

The equation for a hyperbolic paraboloid, Fig. 12-5, is expressed as

$$z = \frac{\bar{y}^2}{2h_2} - \frac{\bar{x}^2}{2h_1} \quad (12-13)$$

where h_1 and h_2 are constants. At $\bar{x} = 0$, or alternatively at $\bar{y} = 0$, this equation reduces to a parabola. At z equal to a constant larger than zero, Eq. (12-13) reduces to a hyperbola. At $z = 0$, Eq. (12-13) reduces to an expression (Kelkar and Sewell 1987) that defines the relationship between two straight lines. These straight lines can be generated from a set of new axes, x and y , that are oriented, Fig. 12-6, with respect to the $\bar{x} - \bar{y}$ plane by angle

$$\tan \psi = \sqrt{\frac{h_2}{h_1}}. \quad (12-14)$$

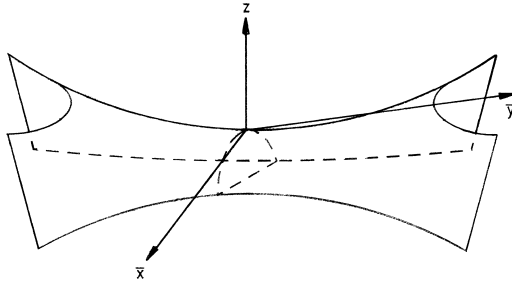


Figure 12-5. Hyperbolic paraboloid shell.

For a rectangular shell, we let $h_1 = h_2 = h$, and the equation for the surface (Billington 1982) becomes

$$z = \frac{h}{ab} xy. \quad (12-15)$$

The load $\bar{p}_z$ in Eq. (12-12) is the projected load over the surface and is normally taken as a constant. Substituting Eq. (12-15) into Eq. (12-12) and letting $\bar{p}_z = p$ gives

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{ab}{2h} p. \quad (12-16)$$

From Eq. (12-9) we get the shear forces as

$$N_{xy} = -\frac{ab}{2h} p \quad (12-17)$$

Integrating Eq. (12-16) gives

$$F = \frac{ab}{2h} pxy + C_1(x) + C_2(y) \quad (12-18)$$

and from Eqs. (12-10) and (12-11) we get forces $\bar{N}_x$ and $\bar{N}_y$ as

$$\bar{N}_x = \frac{\partial^2 C(y)}{\partial y^2} \quad (12-19)$$

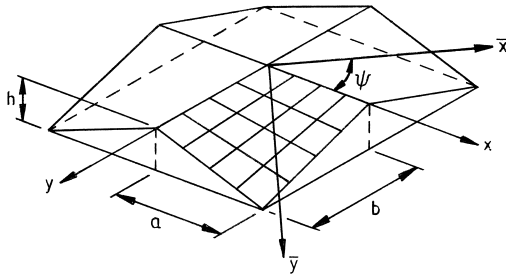


Figure 12-6. Axes orientation for a hyperbolic paraboloid shell.

$$\bar{N}_y = \frac{\partial^2 C(x)}{\partial x^2}. \quad (12-20)$$

Most roof structures have either free edges or edges supported by beams that cannot resist any forces in their weak axis. For these structures, it is customary to assume $N_x = N_y = 0$ throughout the shell by letting C_1 and $C_2 = 0$. This assumes that the only stress in the structure is shear and is given by Eq. (12-17) as

$$\begin{aligned} N_{xy} &= \frac{-ab}{2h} p \\ &= \frac{-ab}{2h} p_z \frac{\sin \beta}{\cos \alpha \cos \phi}. \end{aligned} \quad (12-21)$$

The total vertical force on the structure due to dead and live loads is transferred to the supports through shear stress at the edges given by Eq. (12-21). The allowable shear stress is based on the critical shearing stress of a hyperbolic paraboloid which is approximately the same (Kollar and Dulacska 1984) as the critical shearing stress of a plate. The critical shearing stress of a plate is approximated (Timoshenko and Gere 1961) by the equation

$$\tau_{cr} = \frac{\pi^2 E}{12(1 - \mu^2)(b/t)^2} [5.35 + 4(b/a)^2] \quad (12-22)$$

where b is the small length of the shell.

When the hyperbolic paraboloid is in the shape of a deep arch, Fig. 12-1, the loads are transferred in one direction only. In this case, the forces in the arches are determined by any Structural Analysis method.

Problems

12-4 Derive Eq. (12-15).

12-5 What is the critical buckling shearing stress in a reinforced concrete hyperbolic paraboloid roof, Fig. 12-6, if $a = 25$ ft, $b = 20$ ft, $h = 6$ ft, $t = 4$ inches, $E = 3,100$ ksi, and $\mu = 0.15$?

12-3 Elliptic Paraboloid Shells

The equation for an elliptic paraboloid, Fig. 12-7, is given by

$$z = \frac{x^2}{2h_1} + \frac{y^2}{2h_2}. \quad (12-23)$$

When $x = \text{constant}$, or alternatively when $y = \text{constant}$, Eq. (12-23) becomes a parabola. When z is a constant, the equation becomes an ellipse. Substituting Eq. (12-23) into Eq. (12-12) and letting $\bar{p}_z = p$ gives

$$\frac{\partial^2 F}{\partial y^2} \frac{1}{h_1} + \frac{\partial^2 F}{\partial x^2} \frac{1}{h_2} = p. \quad (12-24)$$

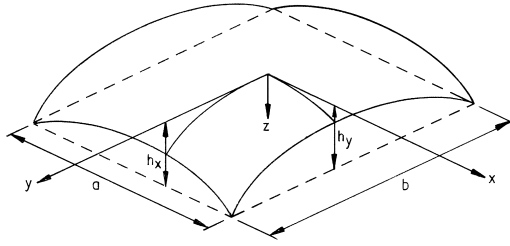


Figure 12-7. Elliptic paraboloid shell.

For many elliptic paraboloid roofs, the edges are either free or supported by beams that cannot support any loads in their weak axes. Thus, N_x and N_y are zero at the boundary conditions. Solution of Eq. (12-24) is achieved by expressing F in a Fourier Series that vanishes at the boundary conditions and is of the form

$$F(x, y) = \sum_{n=1}^{\infty} f_n(x) \cos \frac{n\pi y}{b} \quad n = 1, 3, 5, \dots \quad (12-25)$$

The load p can also be expressed as

$$p = \sum_{n=1}^{\infty} p_n(x) \cos \frac{n\pi y}{b} \quad (12-26)$$

where

$$p_n(x) = \frac{4p}{b} \int_0^{b/2} \cos \frac{n\pi y}{b} dy$$

and Eq. (12-26) becomes

$$p = \sum_{n=1}^{\infty} (-1)^{(n+1)/2} \frac{4p}{\pi n} \cos \frac{n\pi y}{b}. \quad (12-27)$$

Substituting Eqs. (12-25) and (12-27) into Eq. (12-24) yields

$$\frac{d^2 f_n}{dx^2} - k_n^2 f_n = (-1)^{(n+1)/2} \frac{4p}{n\pi} h_2 \quad (12-28)$$

where

$$k_n^2 = \frac{n^2 \pi^2}{b^2} \frac{h_2}{h_1}.$$

Solving this equation for the homogeneous and particular solutions gives

$$f_n(x) = A \sinh kx + B \cosh kx + (-1)^{(n+1)/2} \frac{4pb^2}{n^3 \pi^3} h_1 \quad (12-29)$$

Due to symmetry of loads, $A = 0$. Also, at $x = a/2$, $F = 0$. Thus, Eq. (12-29) can be solved for B as

$$B = (-1)^{(n+1)/2} \frac{4pb^2 h_1}{n^3 \pi^3 \cosh k_n a/2}.$$

With B known, Eqs. (12-29) and (12-25) are solved and the values of N_x , N_y , and N_{xy} are obtained from Eqs. (12-4) and (12-9).

The allowable buckling load, p_{cr} , for elliptic paraboloid shells can be approximated (Kollar and Dulacska 1984) by

$$p_{cr} = 0.366 \frac{Et^2}{2} \left(\frac{1}{R_1^2} + \frac{1}{R_2^2} \right). \quad (12-30)$$

Problems

12-6 Derive Eq. (12-24).

12-7 Determine the values of N_x , N_y , and N_{xy} from Eqs. (12-4) and (12-9) using the expression given by Eqs. (12-25) and (12-29).

12-4 Folded Plates

Folded plate roofs, Fig. 12-8, and bottom hoppers of rectangular storage tanks can be analyzed using the folded plate theory. Methods of analyzing various folded structures vary greatly depending on the needed degree of accuracy. Usually a preliminary analysis is performed first that is based on equilibrium

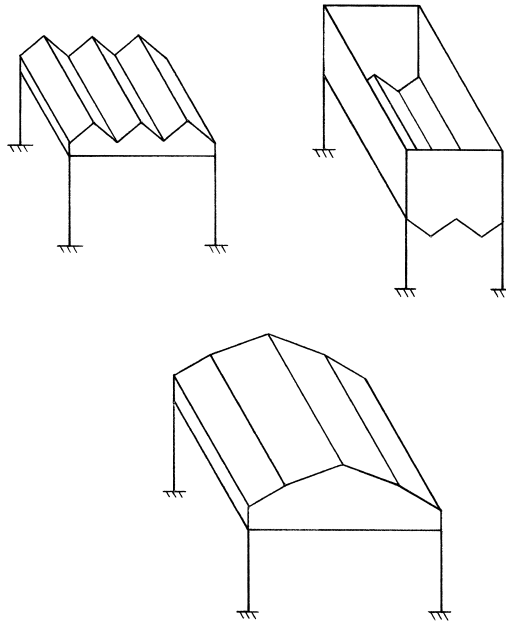


Figure 12-8. Folded plate structures.

considerations. Then, a more elaborate analysis is conducted that takes into consideration the rotation and deflection compatibility of the various components. The following assumptions are normally made when analyzing folded plates

1. The plate material is elastic, isotropic, and homogeneous.
2. The cross section of the plate is constant throughout its span.
3. The supporting end members have infinite stiffness in their own planes and are flexible normal to their own planes.
4. The plates carry loads transversely by bending normal to their planes.
5. The distribution of all loads in the longitudinal direction is the same on all plates.
6. The plates carry loads longitudinally by bending within their planes.
7. Longitudinal stresses vary linearly over the depth of each plate.
8. The torsional stiffness of the plates normal to their own planes is zero.
9. Displacements due to forces other than bending moments are neglected.

The preliminary analysis of folded plates proceeds as follows:

a. Calculate the transverse bending moment for each panel based on an assumed dead weight and applied loads. This is accomplished by assuming the edges of the panels, Fig. 12-9a, to be continuously supported in the transverse direction, Fig. 12-9b. The moments are determined by any Structural Analysis method. The reactions in the transverse supports are then determined.

b. Since the transverse supports are fictitious, the reactions must be eliminated by calculating their equivalent in-plane forces in the adjacent panels as illustrated in Fig. 12-9c.

Steps a and b can be combined into one step by using Matrix Analysis of a frame, taking into consideration axial forces.

c. Calculate the longitudinal bending stresses in each panel due to the in-plane forces from step b. The panels are assumed simply supported at the ends with maximum bending moment in the middle of the span.

d. The longitudinal bending stresses calculated in step c for adjacent panels will be different at the edge intersection because the loads in each panel are different. Thus, it is necessary to apply shearing forces, Fig. 12-9d at the edges of connecting panels in order to have the longitudinal stresses at a given edge in equilibrium. The shearing forces are assumed parabolic in distribution and their magnitude is determined by solving a set of simultaneous equations.

e. The total stress in each panel is determined by combining the stresses determined in steps a, c, and d.

The method discussed in steps a through e satisfies the equilibrium equations across the edges. Compatibility of the deflections of adjacent panels at an edge due to in-plane loads was not considered. Neither was the transverse rotation of adjacent panels at the edges. These two conditions will have to be satisfied in order to complete the analysis of a folded plate.

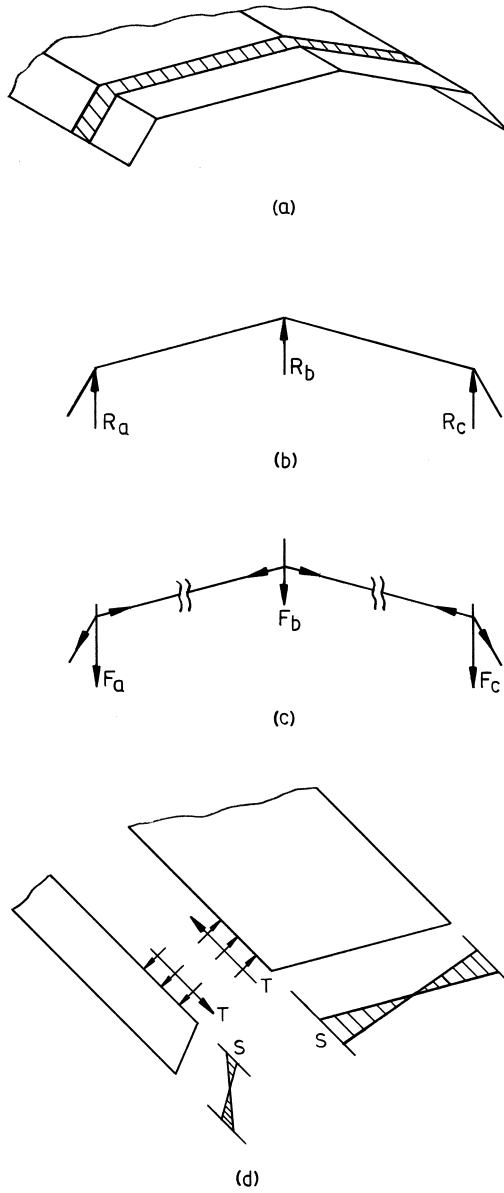


Figure 12-9. Load distribution in a folded plate.

The stresses obtained in steps a through e must be adjusted to take into consideration the deflection and rotation compatibility at each edge. We begin the derivation of the deflection expression by noticing that the shape of the moment diagram due to in-plane loads in the panels is parabolic as shown in Fig. 12-10a and is given by

$$M_x = PLx/2 - Px^2/2 \quad (12-31)$$

The distribution of the shear forces T_x must be assumed triangular in shape, Fig. 12-10, in order for the resulting parabolic moment to be compatible with the load bending moment. Thus, the shear force at any point x in Fig. 12-10b is expressed as

$$T_x = T_0(1 - 2x/L)$$

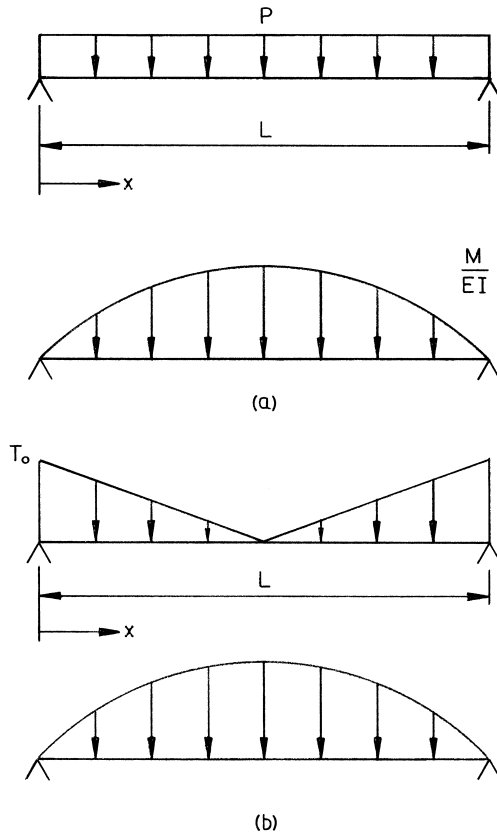


Figure 12-10. Shear and moment distribution due to in-plane loads in folded plates.

and the moment diagram is

$$\begin{aligned} M_x &= \int (d/2)(T_x)(L/2 - x)dx \\ &= (d/2)(T_o)(Lx/2 - x^2 + 2x^3/3L) \end{aligned}$$

or letting $T = (T_o)(L/2)(1/2)$

$$M_x = (2)(d)(T)(x/2 - x^2/L + 2x^3/3L^2). \quad (12-32)$$

This parabolic equation has a similar shape as Eq. (12-31). Hence, the expression for deflection of each panel can be calculated from the combination of Eqs. (12-31) and (12-32). In order to simplify the deflection calculations, the bending moments are expressed in terms of stresses. Referring to Fig. 12-11,

$$\frac{1}{R} = \frac{-\varepsilon_t + \varepsilon_b}{d}.$$

Also, from Eq. (1-6)

$$\chi = 1/R = \frac{d^2y}{dx^2}.$$

Hence, the equation

$$\frac{d^2y}{dx^2} = M/EI$$

becomes

$$M/EI = \frac{-\varepsilon_t + \varepsilon_b}{d}.$$

where d is depth of panel

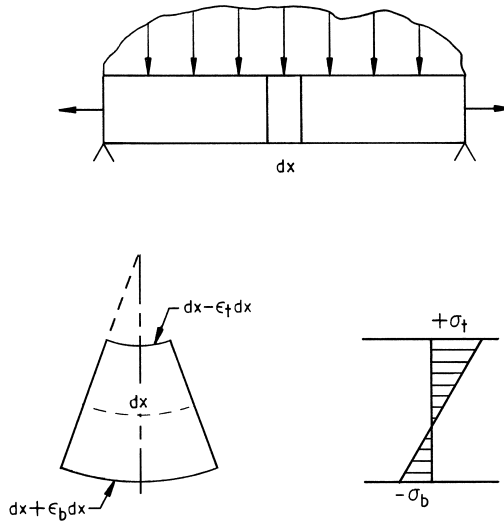


Figure 12-11. Stress distribution due to bending moment and in-plane forces.

Defining

$$\varepsilon = \sigma/E$$

gives

$$\frac{-\sigma_t + \sigma_b}{dE} = M/EI.$$

The deflection at the centerline of the panels can be determined from Structural Analysis methods by using Fig. 12-12. This gives

$$w = \frac{1}{9.6} (L^2) \left(\frac{-\sigma_t + \sigma_b}{dE} \right). \quad (12-33)$$

In this equation, the tensile stress is entered as a positive and the compressive stress as a negative quantity.

From the calculated deflections at each edge, the rotation of each panel is determined from geometry. The change in rotation difference of two adjacent panels at a given edge due to applied external forces is designated by θ_F .

The rotations θ_F at each of the internal edges are due to the fictitious hinges inserted in the structure in order to find the in-plane loads in the panels. These rotations must be eliminated as the edges in the actual structure are rigid. One method of eliminating the rotations θ_F is to apply correction moments at each of the inner edges as shown in Fig. 12-13. Accordingly, the procedure needed to satisfy the rotation compatibility of the panels is as follows

1. From the stresses obtained in step e above, calculate the deflection of each panel and then calculate the net rotation θ_F at each of the inner edges.

2. Apply a correction moment, M_x , at each of the internal edges and calculate the rotation, θ'_{M_x} , at all the inner edges and the corresponding reactions. The shape of these reaction forces along the length of the panel must be the same as the shape of

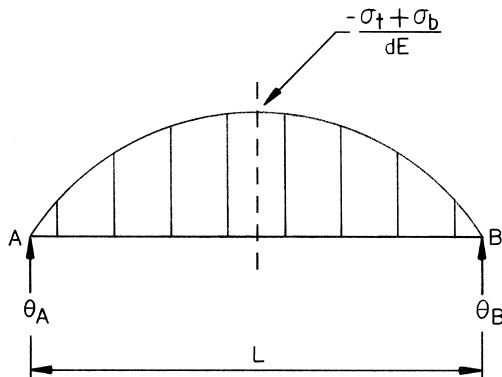


Figure 12-12. Conjugate beam.

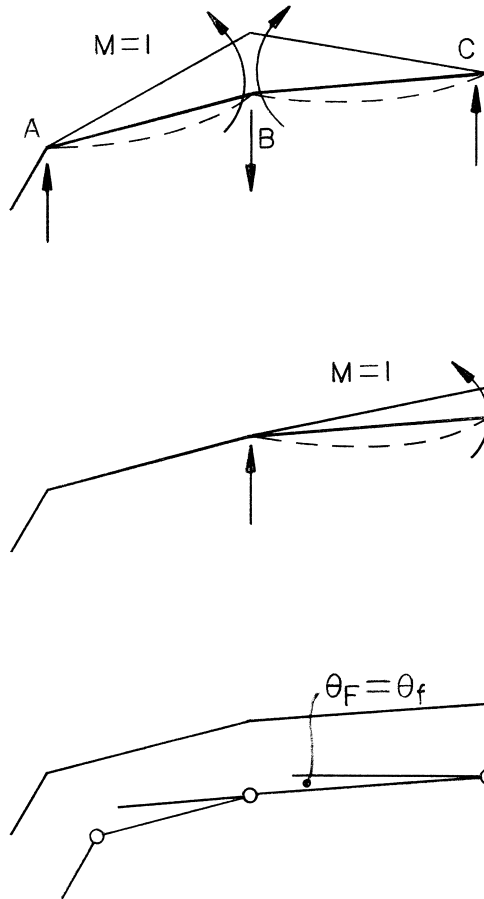


Figure 12-13. Moment correction.

the rotation. Since the shape of the rotation due to Mx must be the same as the rotation due to applied loads, it follows that a parabolic shape is to be selected. However, an approximate shape that is easier to use is of the form

$$P_x = P_o \sin \frac{\pi x}{L}.$$

The expression for the moment is of the form

$$Mx = M_o \sin \frac{\pi x}{L}$$

and from the equations

$$\frac{d^2 y}{dx^2} = \frac{-M}{EI}$$

and

$$\frac{d^4 y}{dx^4} = \frac{-P}{EI}$$

we get

$$M_x = P_0 \frac{L}{\pi^2} \sin \frac{\pi x}{L}. \quad (12-34)$$

3. Resolve the reactions to in-plane loads in the panels and use steps b through e to calculate stresses.

4. Then calculate deflections and rotations, θ''_{Mx} , at all inner edges. The expression for the deflection of a sinusoidal load is given by

$$w = \frac{L^2}{\pi^2} \left(\frac{-\sigma_t + \sigma_b}{dE} \right) \quad (12-35)$$

5. The net rotation at the selected inner edge is then given by

$$\theta_{Mx} = \theta'_{Mx} + \theta''_{Mx}$$

6. Steps 2 through 5 are repeated for each inner edge.

7. Since the net rotation of the panels at each edge is zero, a number of equations can be written as follows:

$$\begin{bmatrix} \theta_{F1} + k_1 \theta_{Mx11} + k_2 \theta_{Mx12} + k_3 \theta_{Mx13} + \cdots = 0 \\ \theta_{F2} + k_1 \theta_{Mx21} + k_2 \theta_{Mx22} + k_3 \theta_{Mx23} + \cdots = 0 \\ \vdots \\ \theta_{Fn} + k_1 \theta_{Mxn1} + k_2 \theta_{Mxn2} + k_3 \theta_{Mxn3} + \cdots = 0 \end{bmatrix} \quad (12-36)$$

where

θ_{F1} = rotation of edge 1 due to applied loads;

θ_{Mx11} = rotation of edge 1 due to moment M_x applied at edge 1;

θ_{Mx12} = rotation of edge 1 due to moment M_x applied at edge 2;

K_i = constants to be determined.

The final stresses are given by

$$\begin{bmatrix} S_1 = S_{1p} + k_1 S_{11} + k_2 S_{12} + \cdots \\ S_2 = S_{2p} + k_1 S_{21} + k_2 S_{22} + \cdots \\ \vdots \\ S_n = S_{np} + k_1 S_{n1} + k_2 S_{n2} + \cdots \end{bmatrix} \quad (12-37)$$

12-5 Barrel Roofs

The governing differential equation for the bending of barrel roofs is derived in a similar manner as that of cylindrical shells in Chapter 10. The shell segment in

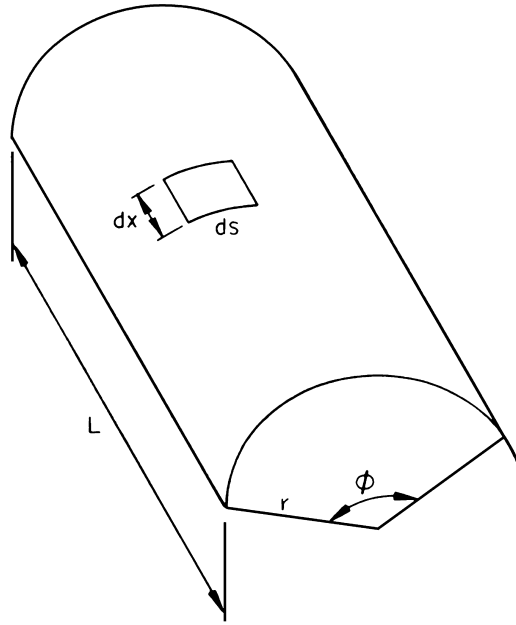


Figure 12-14. Barrel roof.

Fig. 12-14 is subjected to uniform dead and live loads in the z -direction only. Equations for the equilibrium of the element (Gibson 1968) in the x -, y -, and z -directions are derived first. Then expressions are obtained for the stress-strain relationship. The relationship between the various deflections in the x -, y -, and z -axes are also determined from the geometry of the shell section. By a series of combinations and substitutions, and assuming a Poisson's ratio of zero, the equations are simplified into one governing equation of the form

$$r^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{r^2 \partial \phi^2} \right)^4 w + \frac{12}{t^2} \frac{\partial^4 w}{\partial x^4} = \frac{12r^2}{Et^3} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{r^2 \partial \phi^2} \right)^2 p. \quad (12-38)$$

The homogeneous and particular solutions of this equation, taking advantage of loads and boundary conditions (Chattarjee 1971), yields an expression for the deflection with eight constants of integration. These constants are obtained from the boundary conditions of the roof.

The solutions of Eq. (12-38) as well as other corresponding moments and forces are found in many references. The ASCE Manual (ASCE 1960) gives comprehensive theoretical and tabular values for various boundary conditions. Gibson (Gibson 1968), Chatterjee (Chattarjee 1971), and Billington (Billington 1982) give many solved examples using theoretical and tabular values.

Buckling of Cylindrical Shells

13-1 Basic Equations

Cylindrical Shells subjected to compressive forces, Fig. 13-1, must be evaluated in accordance with one of the buckling theories of shells. Two approaches for developing the buckling equations will be discussed. The first is that of Sturm (Sturm 1941) which is well suited for designing cylindrical shells at various temperatures using actual stress–strain curves as discussed in Section 13-5. This approach is used in many pressure vessel codes for the design of cylindrical shells.

The second approach for analyzing buckling of cylindrical shells is that of Donnell (Gerard 1962). This method is discussed in Section 13-6 and is used extensively in the aerospace industry.

We begin Sturm's derivation by taking an infinitesimal element of a cylindrical shell with applied forces and moments as shown in Fig. 13-2. The assumptions made in deriving the pertinent differential equations are

1. The cylinder is round before buckling.
2. The thickness is constant throughout the cylinder.
3. The material is isotropic, elastic, and homogeneous.
4. The thickness is small compared to the radius.
5. Radial stress is negligible compared to the circumferential and longitudinal stresses.

Summation of forces and moments in the x -, y -, and z -directions results in the following six equations of equilibrium:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{\theta x}}{\partial y} - Q_x \frac{\partial^2 w}{\partial x^2} = 0 \quad (13-1)$$

$$\frac{\partial N_{\theta}}{\partial y} + \frac{\partial N_{x\theta}}{\partial x} + Q_{\theta} \frac{\partial \psi}{\partial y} = 0 \quad (13-2)$$

$$\frac{\partial Q_{\theta}}{\partial y} + \frac{\partial Q_x}{\partial x} = p + N_{\theta} \frac{\partial \psi}{\partial y} - N_x \frac{\partial^2 w}{\partial x^2} - N_{x\theta} \frac{\partial^2 w}{\partial y \partial x} - N_{\theta x} \frac{\partial^2 w}{\partial x \partial y} \quad (13-3)$$

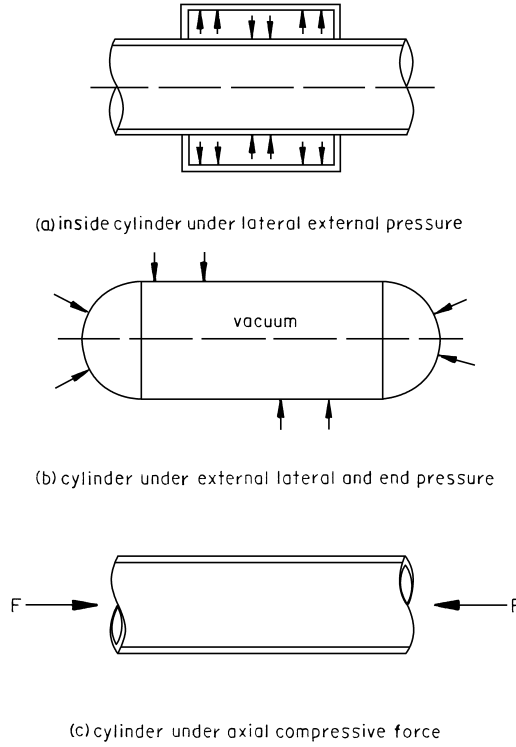


Figure 13-1. Compressive forces in cylindrical shells.

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{\theta x}}{\partial y} \quad (13-4)$$

$$Q_\theta = \frac{\partial M_\theta}{\partial y} + \frac{\partial M_{x\theta}}{\partial x} \quad (13-5)$$

$$N_{\theta x} - N_{x\theta} + M_{\theta x} \frac{\partial \psi}{\partial y} + M_{x\theta} \frac{\partial^2 w}{\partial x^2} = 0 \quad (13-6)$$

where w is the deflection in the z -direction.

The above equations cannot be solved directly because there are more unknowns than available equations. Accordingly, additional equations are needed. We can utilize the stress-strain relationship of Eq. (1-14) and rewrite it in terms of force-strain relationship as

$$N_x = \frac{Et}{1 - \mu^2} (\epsilon_x + \mu \epsilon_\theta) \quad (13-7)$$

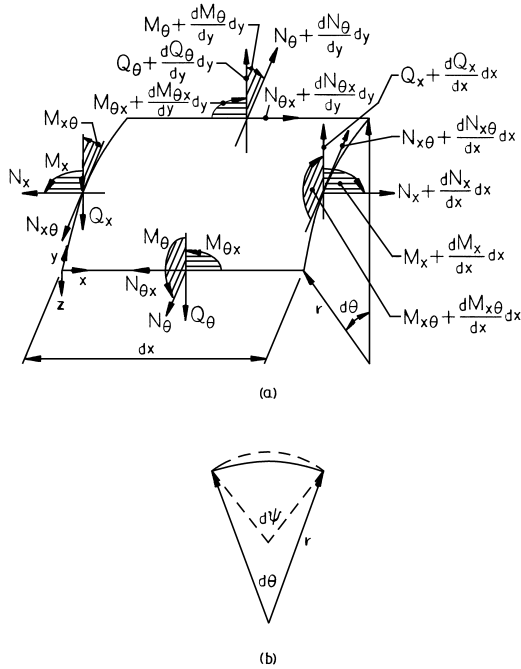


Figure 13-2. Infinitesimal element of a cylindrical shell.

$$N_{\theta} = \frac{Et}{1 - \mu^2} (\epsilon_{\theta} + \mu \epsilon_x) \quad (13-8)$$

$$N_{\theta x} = \frac{Et}{2(1 + \mu)} \gamma_{\theta x} \quad (13-9)$$

where ϵ_x and ϵ_{θ} are the strains in the x - and y -directions and $\gamma_{\theta x}$ is the shearing strain.

Similarly, the moment–deflection equations are expressed as

$$M_x = -D \left(\frac{\partial^2 w}{\partial x^2} + \frac{\mu}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \mu \frac{w}{r^2} \right) \quad (13-10)$$

$$M_{\theta} = -D \left(\mu \frac{\partial^2 w}{\partial x^2} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + \frac{w}{r^2} \right) \quad (13-11)$$

$$M_{\theta x} = -D(1 - \mu) \frac{1}{r} \frac{\partial^2 w}{\partial \theta \partial x}. \quad (13-12)$$

The relationship between strain and deformation is given by

$$\epsilon_x = \frac{\partial \mu}{\partial x} \quad (13-13)$$

$$\varepsilon_{\theta} = \frac{\partial v}{\partial y} + \frac{v}{r} \quad (13-14)$$

$$\gamma_{\theta x} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (13-15)$$

where u and v are the deflections in the x - and y -directions, respectively.

Equation (13-14) is based on the fact that the radial deflection, w , for thin shells produces bending as well as stretching of the middle surface. Hence, from Fig. 13-3.

$$\begin{aligned} \frac{ds_2 - ds_1}{ds_1} &= \frac{(r + w)d\theta + (\partial v / \partial \theta)d\theta - r d\theta}{r d\theta} \\ &= \frac{\partial v}{r \partial \theta} + \frac{w}{r} = \frac{\partial v}{\partial y} + \frac{w}{r}. \end{aligned}$$

The change in angle $\delta\psi$ shown in Fig. 13-2 is expressed as

$$\frac{\partial \psi}{\partial \theta} = 1 - \frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} - \frac{w}{r} + \varepsilon_{\theta}. \quad (13-16)$$

This equation is obtained from Fig. 13-2 where the angle $\delta\psi$ is the total sum of

- (1) $d\theta$ which is the original angle
- (2) $-\frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} d\theta$ which is the change in slope of length ds

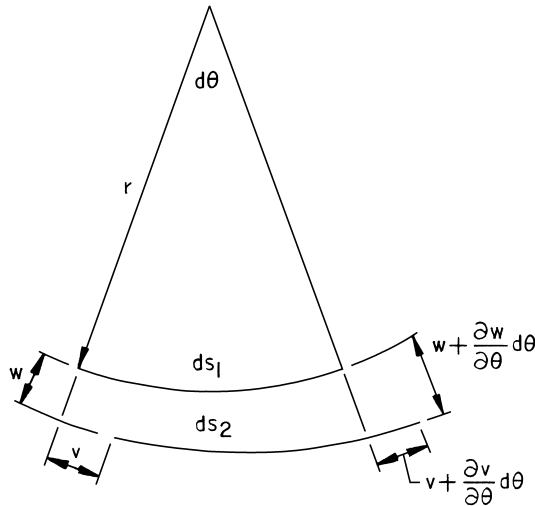


Figure 13-3. Deflected section.

(3) $-\frac{w}{r} d\theta$ which is due to radial deflection

(4) $\varepsilon_\theta d\theta$ which is due to circumferential strain.

The derivative of these four expressions results in Eq. (13-16).

Assuming $M_{xy} = M_{yx}$, the above 16 equations contain the following unknowns: N_x , N_θ , $N_{x\theta}$, $N_{\theta x}$, Q_x , Q_θ , M_x , M_θ , $M_{x\theta}$, ε_x , ε_θ , $\gamma_{x\theta}$, w , u , v , and ψ . These 16 equations can be reduced to four by the following various substitutions.

From Eqs. (13-13), (13-14), and (13-15) we get

$$\frac{\partial^2 \varepsilon_\theta}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial y^2} - \frac{\partial^2 \gamma_{\theta x}}{\partial x \partial y} - \frac{1}{r} \frac{\partial^2 w}{\partial x^2} = 0. \quad (13-17)$$

Substituting Eqs. (13-7), (13-8), and (13-9) into Eq. (13-17) gives

$$\frac{\partial^2 N_\theta}{\partial x^2} - \mu \frac{\partial^2 N_x}{\partial x^2} + \frac{1}{r^2} \frac{\partial^2 N_x}{\partial \theta^2} - \frac{\mu}{r^2} \frac{\partial^2 N_\theta}{\partial \theta^2} - \frac{2(1+\mu)}{r} \frac{\partial^2 N_{\theta x}}{\partial \theta \partial x} = \frac{Et}{r} \frac{\partial^2 w}{\partial x^2}. \quad (13-18)$$

The shearing forces in Eqs. (13-1) and (13-2) are eliminated by combining these two equations with Eqs. (13-4), (13-5), and (13-6). This gives

$$\frac{\partial^2 N_\theta}{\partial y^2} - \frac{\partial^2 N_x}{\partial x^2} + \frac{1+\varepsilon_\theta}{r} \frac{\partial^2 M_\theta}{\partial y^2} + \frac{2(1+\varepsilon_\theta)}{r} \frac{\partial^2 M_{\theta x}}{\partial y \partial x} = 0. \quad (13-19)$$

Equations (13-4) and (13-5) are combined with Eq. (13-3) to give

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{\theta x}}{\partial y \partial x} + \frac{\partial^2 M_\theta}{\partial y^2} = p + N_\theta \frac{\partial \psi}{\partial y} - N_x \frac{\partial^2 w}{\partial x^2} - N_{x\theta} \frac{\partial^2 w}{\partial y \partial x} - N_{\theta x} \frac{\partial^2 w}{\partial y \partial x}. \quad (13-20)$$

Substituting Eqs. (13-10), (13-11), and (13-12) into Eq. (13-20) results in the first of the four basic equations we are seeking:

$$\begin{aligned} & -D \left[\frac{\partial^4 w}{\partial x^4} + \frac{\mu}{r^2} \frac{\partial^2 w}{\partial x^2} + \frac{2}{r^2} \frac{\partial^4 w}{\partial \theta^2 \partial x^2} + \frac{1}{r^4} \frac{\partial^4 w}{\partial \theta^4} + \frac{1}{r^4} \frac{\partial^2 w}{\partial \theta^2} \right] \\ & = p + N_\theta \frac{1}{r} \frac{\partial \psi}{\partial \theta} - N_x \frac{\partial^2 w}{\partial x^2} - N_{x\theta} \frac{1}{r} \frac{\partial^2 w}{\partial \theta \partial x} - N_{\theta x} \frac{1}{r} \frac{\partial^2 w}{\partial \theta \partial x}. \end{aligned} \quad (13-21)$$

The second basic equation is obtained by combining Eqs. (13-11), (13-12), and (13-19):

$$\frac{1}{r^2} \frac{\partial^2 N_\theta}{\partial \theta^2} - \frac{\partial^2 N_x}{\partial x^2} - \frac{(1+\varepsilon_\theta)}{r} D \left[\frac{1}{r^4} \frac{\partial^4 w}{\partial \theta^4} + \frac{1}{r^4} \frac{\partial^2 w}{\partial \theta^2} + \frac{(2-\mu)}{r^2} \frac{\partial^4 w}{\partial \theta^2 \partial x^2} \right] = 0. \quad (13-22)$$

Combining Eqs. (13-18) and (13-1) yields the third basic equation:

$$\frac{\partial^2 N_\theta}{\partial x^2} + (2+\mu) \frac{\partial^2 N_x}{\partial x^2} + \frac{1}{r^2} \frac{\partial^2 N_x}{\partial \theta^2} - \frac{\mu}{r^2} \frac{\partial^2 N_\theta}{\partial \theta^2} = \frac{Et}{r} \frac{\partial^2 w}{\partial x^2}. \quad (13-23)$$

Solving for ε_θ from Eqs. (13-7) and (13-8) and differentiating twice with respect to x gives the fourth basic equation:

$$\frac{\partial^2 N_\theta}{\partial x^2} - \mu \frac{\partial^2 N_x}{\partial x^2} = \frac{Et}{r} \left(\frac{\partial^3 v}{\partial x^2 \partial \theta} + \frac{\partial^2 w}{\partial x^2} \right). \quad (13-24)$$

Equations (13-21) through (13-24) are the four basic equations needed to develop a solution for the buckling of cylindrical shells.

13-2 Lateral Pressure

When the external pressure is applied only to the side of the cylinder as shown in Fig. 13-1a, then the solution can be obtained as follows. Let

$$N_\theta = -pr + f(x, y) \quad (13-25)$$

where $f(x, y)$ is a function of x and y which expresses the variation of N_θ from the average value. When the deflection, w , is small, then the function $f(x, y)$ is also very small. Similarly, the end force N_x is expressed as

$$N_x = 0 + g(x, y). \quad (13-26)$$

Since $N_{\theta x} = N_{x\theta} = 0$, then

$$N_{\theta x} = 0 + h(x, y) \quad (13-27)$$

$$N_{x\theta} = 0 + j(x, y). \quad (13-28)$$

Substituting Eqs. (13-25) through (13-28) into Eqs. (13-21) through (13-24) and neglecting higher order terms such as

$$g \frac{\partial^2 w}{\partial \theta \partial x}, h \frac{\partial^2 w}{\partial \theta \partial x}, j \frac{\partial^2 w}{\partial \theta \partial x}, \text{ and } f \text{ with terms in } \frac{\partial \psi}{\partial \theta}$$

other than unity results in the following four equations:

$$\begin{aligned} D \left[\frac{\partial^4 w}{\partial x^4} + \frac{\mu}{r^2} \frac{\partial^2 w}{\partial x^2} + \frac{2}{r^2} \frac{\partial^4 w}{\partial \theta^2 \partial x^2} + \frac{1}{r^4} \left(\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^4 w}{\partial \theta^4} \right) \right] + \frac{1}{r} f(x, y) \\ = -p \left[\frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} + \frac{w}{r} + \varepsilon_\theta \right] \end{aligned} \quad (13-29)$$

$$\frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} - \frac{\partial^2 g}{\partial x^2} - \frac{(1 + \varepsilon_\theta)}{r^3} D \left[\frac{1}{r^2} \frac{\partial^4 w}{\partial \theta^4} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} + (2 - \mu) \frac{\partial^4 w}{\partial \theta^2 \partial x^2} \right] = 0 \quad (13-30)$$

$$\frac{\partial^2 f}{\partial x^2} + (2 + \mu) \frac{\partial^2 g}{\partial x^2} + \frac{1}{r^2} \frac{\partial^2 g}{\partial \theta^2} - \frac{\mu}{r^2} \frac{\partial^2 f}{\partial \theta^2} = \frac{Et}{r} \frac{\partial^2 w}{\partial x^2} \quad (13-31)$$

$$\frac{\partial^2 f}{\partial x^2} - \mu \frac{\partial^2 g}{\partial x^2} = \frac{Et}{r} \left(\frac{\partial^3 v}{\partial x^2 \partial \theta} + \frac{\partial^2 w}{\partial x^2} \right). \quad (13-32)$$

Equations (13-29) through (13-32) can be solved for various boundary conditions. For a simply supported cylinder, the following conditions are obtained from Fig. 13-4:

$$\text{at } x = \pm L/2, \quad w = \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial \theta^2} = 0.$$

Also, because of symmetry,

$$\frac{\partial w}{\partial \theta} = 0 \text{ for all values of } \theta \text{ when } x = 0$$

and

$$\frac{\partial w}{\partial \theta} = 0 \text{ for all values of } x \text{ when } \theta = 0.$$

Similarly, $\frac{\partial v}{\partial \theta} = 0$ for all values of θ at $x = \pm L/2$.

These boundary conditions suggest a solution of the form

$$w = A \cos n\theta \cos \frac{\pi x}{L}$$

$$v = B \sin n\theta \cos \frac{\pi x}{L}$$

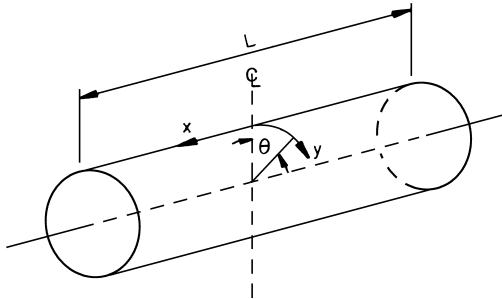


Figure 13-4. Coordinate system in a cylindrical shell.

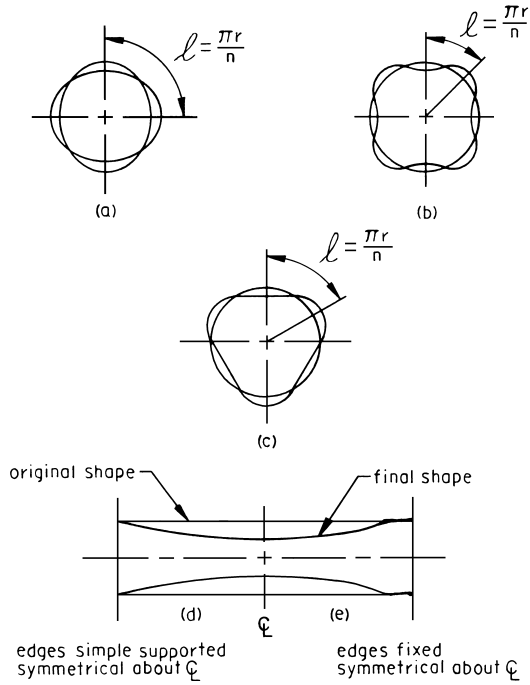


Figure 13-5. Buckling modes of a cylindrical shell.

where n is the number of lobes as defined in Fig. 13-5. Substituting these two expressions into Eqs. (13-29) through (13-32) gives

$$D \left[\frac{\pi^4}{L^4} + \frac{(2n^2 - \mu)\pi^2}{r^2 L^2} - \frac{n^2}{r^4} + \frac{n^4}{r^4} \right] A \cos n\theta \cos \frac{\pi x}{L} = \frac{P}{r} (An^2 + Bn) \cos n\theta \cos \frac{\pi x}{L} - \frac{1}{r} f(x, y) \quad (13-33)$$

$$\frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} - \frac{\partial^2 g}{\partial x^2} = \frac{(1 - \epsilon_\theta)}{r^3} D \left[\frac{n^4}{r^2} - \frac{n^2}{r^2} + (2 - \mu) \frac{n^2 \pi^2}{L^2} \right] A \cos n\theta \cos \frac{\pi x}{L} \quad (13-34)$$

$$(2 + \mu) \frac{\partial^2 g}{\partial x^2} + \frac{1}{r^2} \frac{\partial^2 g}{\partial \theta^2} + \frac{\partial^2 f}{\partial x^2} - \frac{\mu}{r^2} \frac{\partial^2 f}{\partial \theta^2} = \frac{-Et}{r} \frac{\pi^2}{L^2} A \cos n\theta \cos \frac{\pi x}{L} \quad (13-35)$$

$$\frac{\partial^2 f}{\partial x^2} - \mu \frac{\partial^2 g}{\partial x^2} = -\frac{Et}{r} \frac{\pi^2}{L^2} (Bn + A) \cos n\theta \cos \frac{\pi x}{L}. \quad (13-36)$$

From Eq. (13-33) it follows that

$$f(x, y) = C \cos n\theta \cos \frac{\pi x}{L} \quad (13-37)$$

and from Eqs. (13-34) and (13-35) that

$$g(x, y) = G \cos n\theta \cos \frac{\pi x}{L}. \quad (13-38)$$

From Eq. (13-36)

$$Bn + A = \frac{r}{Et} (C - \mu G) \quad (13-39)$$

and the values of C and G are found to be

$$\frac{C}{A} = \frac{Et}{r\alpha^2} - \frac{Et}{r^3(1-\mu^2)} (H-1) \left(1 - \frac{Pr}{Et}\right) \left(\frac{\alpha+1-\mu}{\lambda\alpha}\right) \quad (13-40)$$

$$\frac{G}{A} = \frac{Et}{r\alpha\lambda} + \frac{Et}{r^3(1-\mu^2)} (H-1) \left(1 - \frac{Pr}{Et}\right) \left(\frac{1-\mu(\alpha-1)}{\lambda\alpha}\right) \quad (13-41)$$

where

$$H = n^2[1 + (\lambda-1)(2-\mu)]$$

$$\lambda = \frac{\pi^2 r^2}{n^2 L^2} + 1$$

$$\alpha = \frac{n^2 L^2}{\pi^2 r^2} + 1.$$

From Eq. (13-33),

$$\begin{aligned} & \frac{Et}{r^2 \alpha^2} - \frac{D}{r^4} \frac{\alpha+1+\mu}{\alpha\lambda} (H-1) A \cos n\theta \cos \frac{\pi x}{L} \\ & + \frac{D}{r^4} n^2 (n^2 \lambda^2 - \mu(\lambda-1) - 1) A \cos n\theta \cos \frac{\pi x}{L} \\ & = \frac{p}{r} F A \cos n\theta \cos \frac{\pi x}{L} \end{aligned} \quad (13-42)$$

where

$$\begin{aligned} F &= n^2 - 1 + \frac{1}{\alpha} - \frac{\mu}{\alpha\lambda} - \frac{D}{r^2 Et \alpha \lambda} \\ & \times \left\{ (H-1) \left(1 - \frac{Pr}{Et}\right) [\alpha(1-\mu^2) + (1+\mu^2)] + \alpha + 1 + \mu \right\}. \end{aligned} \quad (13-43)$$

Equation (13-42) indicates that solutions different from zero exist only if

$$p_{cr} = \frac{X + Y - Z}{F} \quad (13-44)$$

where

$$X = \frac{Et}{r\alpha^2}$$

$$Y = \frac{D}{r^3} n^2 [n^2 \lambda^2 - \mu(\lambda - 1) - 1]$$

$$Z = \frac{D}{r^3} \frac{\alpha + 1 + \mu}{\alpha \lambda} (H - 1).$$

Equation (13-44) can be written as

$$p_{cr} = \frac{1}{8} KE \left(\frac{t}{r} \right)^3 \quad (13-45)$$

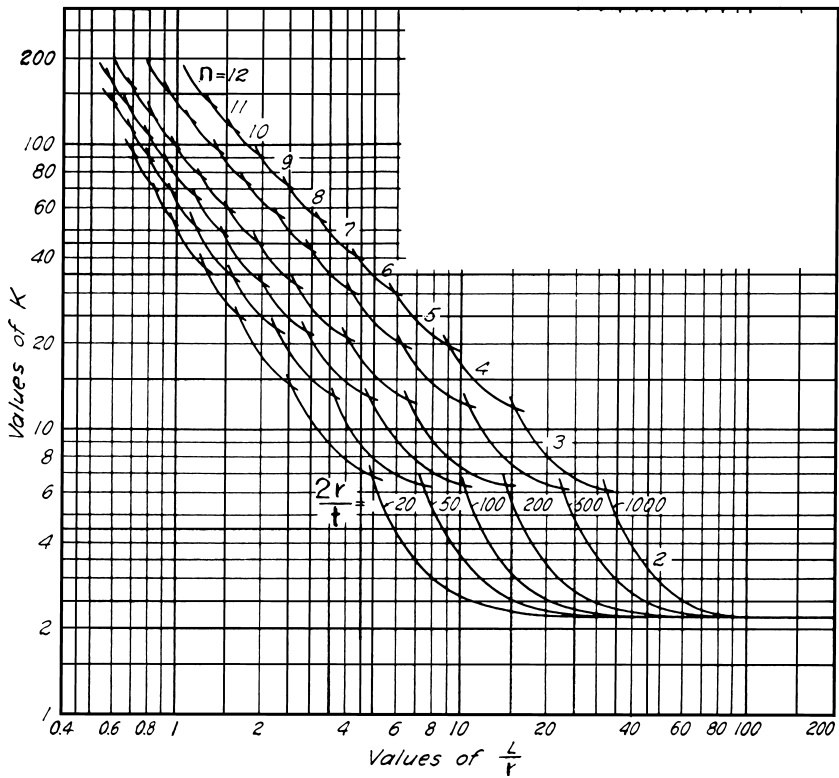


Figure 13-6. Buckling coefficient K for cylinders with pressure on sides only, edges simply supported; $\mu = 0.30$. (Sturm 1941.)

where

$$K = K_1 + 4K_2(r/t)^2 \quad (13-46)$$

$$K_1 = \frac{2}{3} \frac{n^2[n^2\lambda^2 - \mu(\lambda - 1)] - U(H - 1)}{F(1 - \mu^2)} \quad (13-47)$$

$$K_2 = \frac{2}{\alpha^2 F} \quad (13-48)$$

and

$$U = \frac{\alpha + 1 + \mu}{\alpha\lambda}.$$

Equation (13-45) is the basic equation for the buckling of cylindrical shells subjected to lateral pressure. A plot of K in Eq. (13-45) is shown in Fig. 13-6.

Example 13-1

Find the allowable external pressure for the inner cylinder shown in Fig. 13-7. Let $L = 10$ ft, $r_i = 2$ ft, $E = 29,000$ ksi, $t_i = 1/2$ inch, factor of safety ($F.S.$) = 2.5, and $\mu = 0.3$. Assume the inner cylinder to be simply supported.

Solution

$$L/r = 5, \quad 2r/t = 96$$

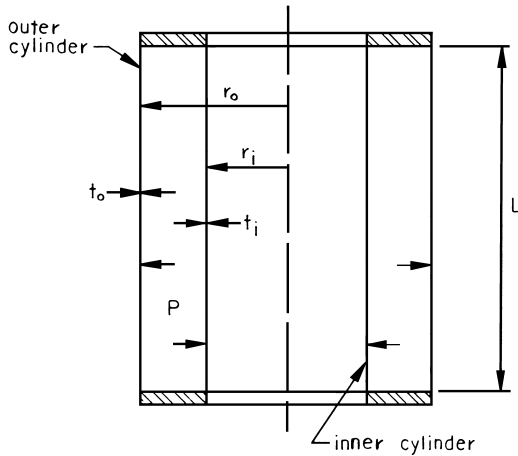


Figure 13-7. Jacketed cylindrical shell.

From Fig. 13-6, $K = 11$. Hence, from Eq. (13-45),

$$\begin{aligned} p_{cr} &= \frac{11}{8} (29,000,000) (0.5/24)^3 \\ &= 360 \text{ psi} \\ p &= p_{cr}/F.S. = 360/2.5 = 144 \text{ psi.} \end{aligned}$$

Equation (13-45) assumes the end of the cylindrical shell to be simply supported. A similar equation can be derived for the case of a cylindrical shell with fixed ends. In this case the slope and deflection at the ends are zero. Proceeding in a similar fashion as for the simply supported case, a buckling equation is obtained. The derivation is more complicated than that for the simply supported cylinder. The resulting buckling equation is the same as Eq. (13-45) with the exception of the value of K . A plot of K for the fixed end condition is shown in Fig. 13-8.

Problems

13-1 Derive Eq. (13-40).

13-2 Derive Eq. (13-42).

13-3 What is the required thickness of the inner cylindrical shell? Let $p = 15$ psi, $L = 20$ ft, $r = 20$ inches, $F.S. = 3.0$, $E = 15,000$ ksi, and $\mu = 0.3$. Assume ends to be fixed.

13-3 Lateral and End Pressure

Many cylindrical shells are subjected to axial forces in the lateral and axial directions, Fig. 13-1b, or to vacuum. The governing equations are very similar to those derived for the lateral condition. For lateral and axial loads, Eq. (13-26) is written as

$$N_x = -\frac{pr}{2} + g(x, y) \quad (13-49)$$

while Eqs. (13-25), (13-27), and (13-28) remain the same. Equation (13-29) becomes

$$\begin{aligned} D \left[\frac{\partial^4 w}{\partial x^4} + \frac{\mu}{r} \frac{\partial^4 w}{\partial x^4} + \frac{2}{r^2} \frac{\partial^4 w}{\partial \theta^2 \partial x^2} + \frac{1}{r^4} \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^4 w}{\partial \theta^4} \right] + \frac{1}{r} f(x, y) \\ = -p \left[\frac{1}{r} \frac{\partial^2 w}{\partial \theta^2} + \frac{w}{r} + \epsilon_\theta \right] - \frac{pr}{2} \frac{\partial^2 w}{\partial x^2} \end{aligned} \quad (13-50)$$

while Eqs. (13-30), (13-31), and (13-32) remain the same. Using the boundary conditions for a simply supported cylinder, the governing Eq. (13-45) can be written as

$$p_{cr} = \frac{1}{8} K' E (t/r)^3 \quad (13-51)$$

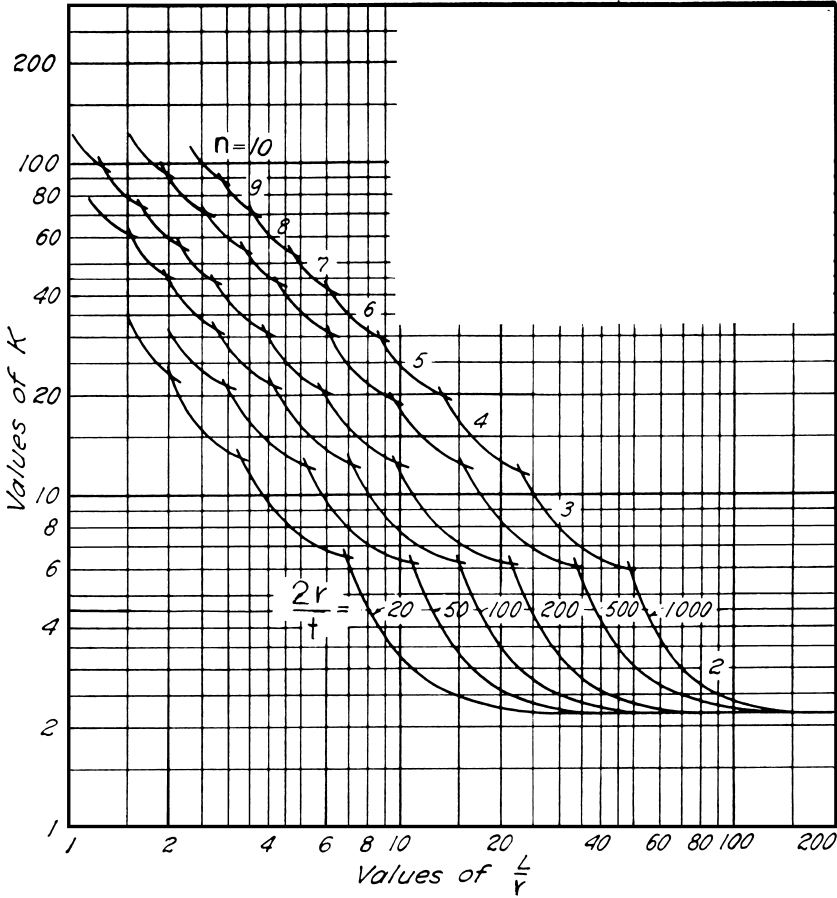


Figure 13-8. Buckling coefficient K for cylinders with pressure on sides only, edges fixed; $\mu = 0.30$. (Sturm 1941.)

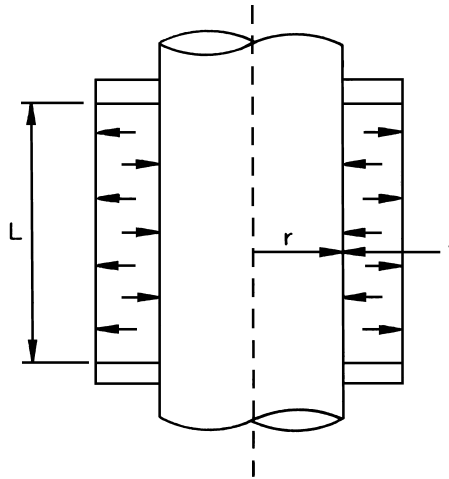
where

$$K' = K'_1 + 4K'_2 (r/t)^2 \quad (13-52)$$

$$K'_1 = K_1 \frac{F}{F + \frac{\pi^2 r^2}{2L^2}}$$

$$K'_2 = K_2 \frac{F}{F + \frac{\pi^2 r^2}{2L^2}}.$$

A plot of Eq. (13-52) for a simply supported cylinder is shown in Fig. 13-9.



Prob. 13-3. Partially jacketed shell.

A similar equation can be derived for a cylinder with fixed ends. The resulting K' value is plotted in Fig. 13-10.

Example 13-2

Determine if the fuel tank shown in Fig. 13-11 is adequate for full vacuum condition. Let $E = 16,000$ ksi, $\mu = 0.3$ and factor of safety ($F.S.$) = 4.

Solution

$$L/r = \frac{96}{24} = 4.00$$

$$2r/t = 192.$$

Assume the ends of the cylinder to be simply supported due to the flexibility of the end plates. From Fig. 13-9, $K' = 20$ and from Eq. (13-51),

$$\begin{aligned} p &= (1/4) \left[\frac{20}{8} \times 16,000,000 (0.25/24)^3 \right] \\ &= 11.3 \text{ psi} < 14.7 \text{ psi. no good.} \end{aligned}$$

For $t = 5/16$ inch, we get $K' = 16$ and $p = 17.66$ psi.

Problems

13-4 Find the required thickness of the inner cylinder shown due to a full vacuum condition, 15 psi, in the inner cylinder. Use Eq. (13-51). Let $E = 29,000$ ksi and $\mu = 0.30$. Use multiples of 1/16 inch in determining the cylinder thickness.

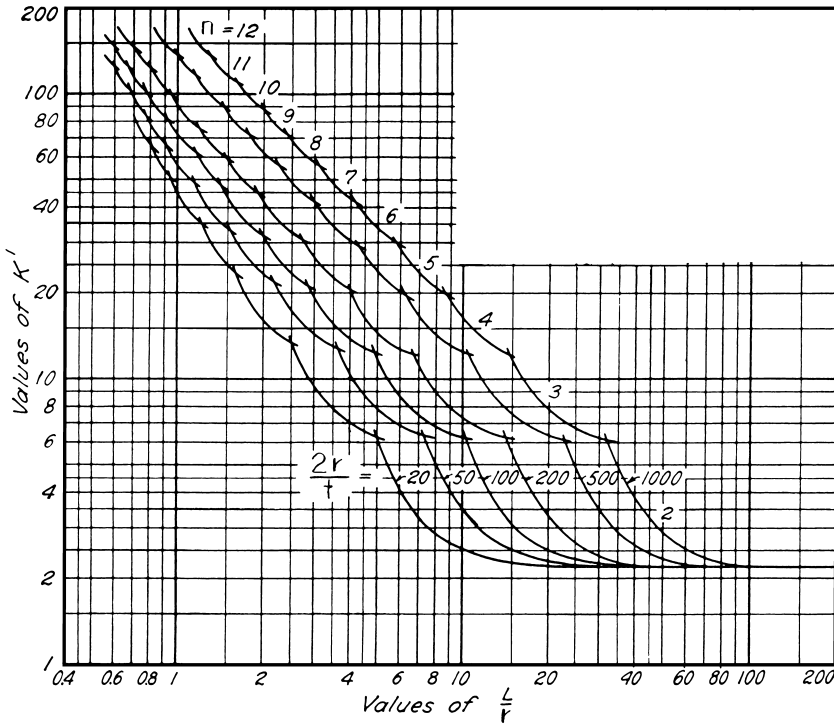


Figure 13-9. Buckling coefficient K' for cylinders with pressure on sides and ends, edges simply supported; $\mu = 0.30$. (Sturm 1941.)

13-5 In Problem 13-4, find the required thickness in the outer cylinder due to a full vacuum condition in the annular jacket space. Use Eq. (13-51) even though the axial force on the outer cylinder is substantially less than that given by Eq. (13-51).

13-6 Solve Problem 13-5 using Eq. (13-45) as a more appropriate equation due to the small end load. What is the difference between this thickness and that obtained in Problem 13-5?

13-4 Axial Compression

When an axial force is applied at the end of a cylinder, the buckling strength is developed as follows. Let W be the applied end load per unit length of circumference. Equations (13-25) and (13-26) become

$$N_\theta = 0 + f(x, y) \quad (13-53)$$

$$N_x = -W + g(x, y). \quad (13-54)$$

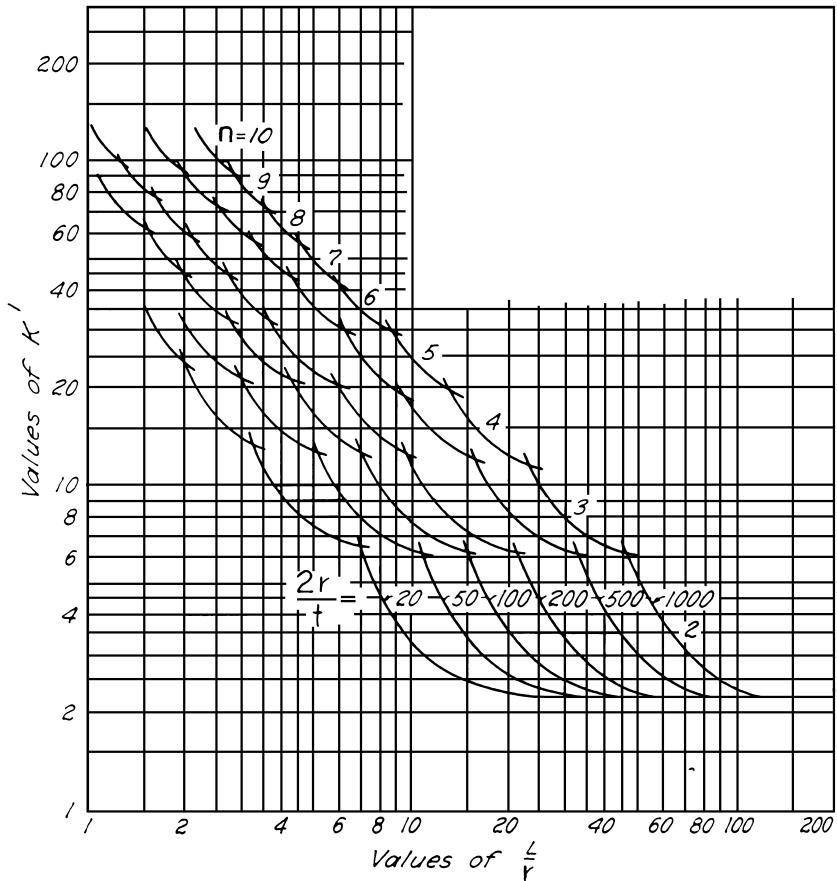


Figure 13-10. Buckling coefficient K' for cylinders with pressure on sides and ends, edges fixed; $\mu = 0.30$. (Sturm 1941.)

For a simply supported cylinder, the buckling equation can be expressed as

$$W_{cr} = \frac{X + Y - Z}{n^2 + V(H - 1)} \quad (13-55)$$

where

$$X = \frac{Et}{\alpha_o \lambda_o}$$

$$Y = \frac{D(\alpha_o - 1)}{r^2} \{ n^2 [n^2 \lambda_o^2 - \mu(\lambda_o - 1)] - 1 \}$$

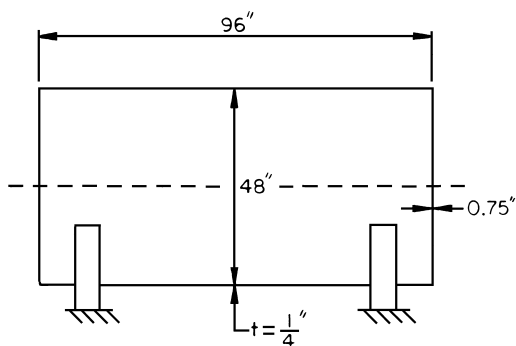
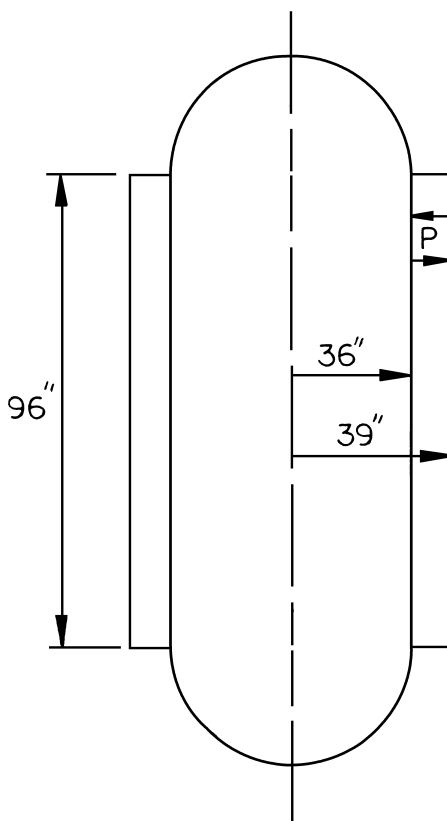


Figure 13-11. Gasoline tank.



Prob. 13-4. Jacketed pressure vessel.

$$Z = \frac{D(\alpha_o - 1)}{r^2} \frac{\alpha_o + 1 + \mu}{\alpha_o \lambda_o} (H - 1)$$

$$V = \frac{\mu D(\alpha + 1 + \mu)}{r E t^2 \lambda_o^2}$$

$$\alpha_o = \frac{n^2 L_o^2}{\pi^2 r^2} + 1$$

$$\lambda_o = \frac{\pi^2 r^2}{n^2 L_o^2} + 1$$

L_o = length of one buckle wave ($L_o \ll L$).

Equation (13-55) can be simplified for design purposes.

Let $\sigma_{cr} = W_{cr}/t$, and for long cylinders with small values of r/t , Eq. (13-55) becomes

$$\sigma_{cr} = 0.6Et/r. \quad (13-56)$$

Tests have shown that for actual cylinders, which have a slight out-of-roundness, the stress magnitude given by Eq. (13-56) tends to be unconservative for large r/t ratios. Figure 13-12 shows the scatter of some of the data obtained from various tests. An empirical equation that defines the lower bound of available test data (ACI 2002) is given by

$$\sigma_{cr} = 0.6CEt/r \quad (13-57)$$

where

$$C = 1 - 0.9(1 - e^{(-1/16)\sqrt{(r/t)}}).$$

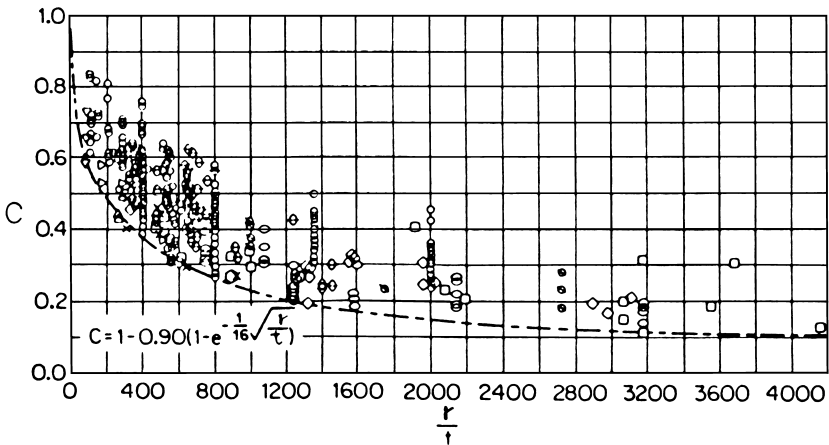


Figure 13-12. Comparison of Eq. (13-56) with test data. (Seide 1981.)

Example 13-3

Determine the thickness of the lower cylinder shown in Fig. 13-13 needed to support a reactor of weight 400,000 lbs. Use Eq. (13-57). Let $E = 30,000$ ksi and factor of safety ($F.S.$) = 4.0.

Solution

Try $t = 0.25$ inch. Then $r/t = 192$.

From Eq. (13-57), $C = 0.48$

and

$$\begin{aligned}\sigma &= (1/4)(0.48 \times 0.6 \times 30,000,000 \times 0.25/48) \\ &= 11,250 \text{ psi.}\end{aligned}$$

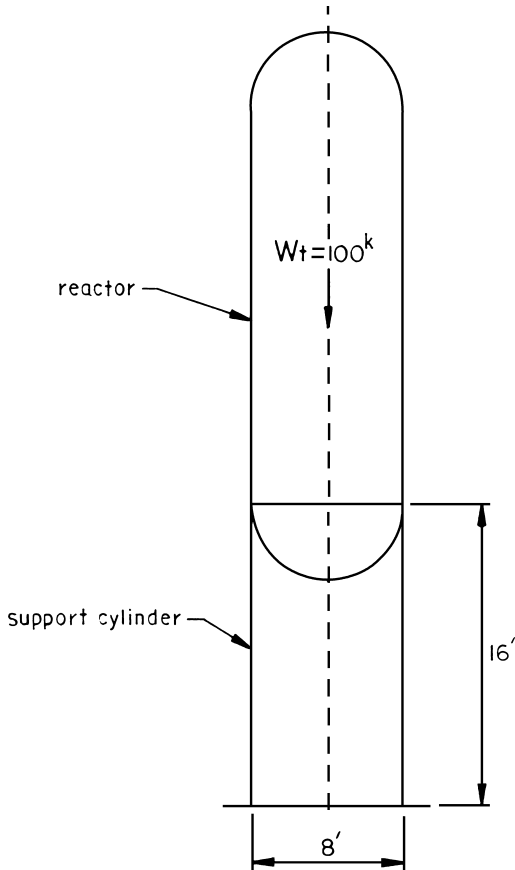
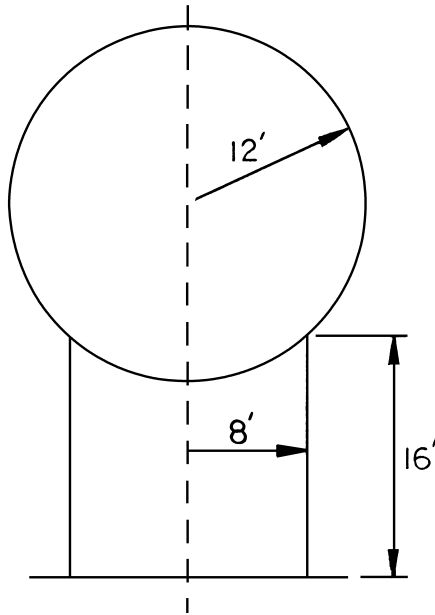


Figure 13-13. Support cylinder.



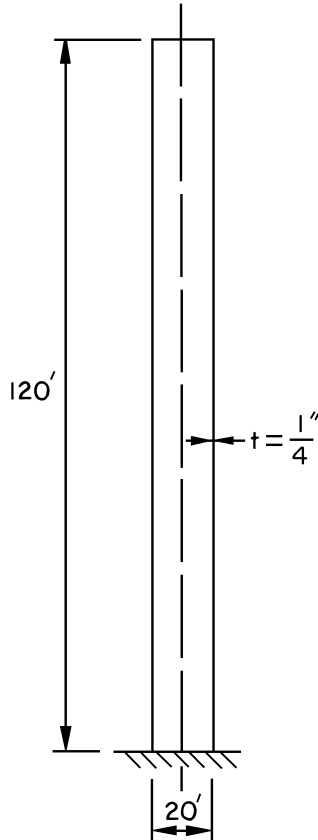
Prob. 13-7. Support cylinder for a spherical fuel tank.

Actual stress is given by

$$= \frac{400,000}{\pi \times 96 \times 0.25} = 5300 \text{ psi.}$$

Theoretically, the thickness can be reduced further. However, consideration must be given to handling and field erection procedures.

It should be noted that the buckling equations derived for lateral pressure, lateral plus axial pressure, and axial forces are based on numerous assumptions and approximations. Experimental tests have shown that these assumptions and approximations tend to make the calculated critical buckling pressure unconservative for large r/t ratios. This is partly due to out-of-roundness of large r/t cylinders caused by such factors as fabrication forming processes, weld shrinkage, and installation tolerances. Accordingly, large factors to safety are usually applied to the calculated buckling pressure when used for design purposes. This is explained further in section 13-6 under Design Equations.



Prob. 13-8. Stand pipe.

Problems

13-7 What is the required thickness of the support cylinder shown. The weight of the spherical tank is 100,000 lbs. Let $E = 30,000$ ksi, $\mu = 0.30$, factor of safety, $(F.S.) = 5.0$. Use increments of $1/16$ inch to determine thickness.

13-8 An oil storage standpipe has a thickness of $1/4"$. Is the thickness adequate for axial compression due to dead weight? Let $E = 30,000$ ksi, $\mu = 0.30$, factor of safety $(F.S.) = 5.0$, and weight of steel = 490 pcf.

13-5 Donnell's Equations

Donnell in 1933 was able to combine Eqs. (13-1) through (13-16) by means of various substitutions to obtain one equation for the buckling of cylindrical shells. The eighth-order differential equation he obtained is a function of deflection

(Gerard 1962) and is expressed as

$$D\nabla^8 w + \frac{Et}{r^2} \frac{\partial^4 w}{\partial x^4} + \nabla^4 \left(N_x \frac{\partial^2}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial \theta} + N_\theta \frac{\partial^2 w}{\partial \theta^2} \right) = 0. \quad (13-58)$$

This equation can be used for various loading conditions. Let us take the simple case of applied lateral pressure, where

$$N_x = N_{xy} = 0 \text{ and } N_\theta = pr = \sigma_{cr} t. \quad (13-59)$$

Equation (13-58) then becomes

$$D\nabla^8 w + \frac{Et}{r^2} \frac{\partial^4 w}{\partial x^4} + \nabla^4 \left(N_\theta \frac{\partial^4 w}{\partial \theta^2} \right) = 0. \quad (13-60)$$

An expression for the deflection that satisfies the simply supported boundary condition of a cylinder can be expressed as

$$w = w_{mm} \sin \frac{m\pi x}{L} \sin \frac{n\pi y}{\pi r}. \quad (13-61)$$

Substituting this expression into Eq. (13-60) gives the nontrivial solution

$$\sigma_{cr} = \frac{\pi^2 KE}{12(1 - \mu^2)} (t/L)^2 \quad (13-62)$$

where

$$K = \frac{(m^2 + \beta^2)^2}{\beta^2} + \frac{12Z^2}{\pi^4 \beta^2 (1 + \beta^2/m^2)^2} \quad (13-63)$$

$$\beta = \frac{nL}{\pi r} \quad (13-64)$$

Z = curvature parameter

$$= \frac{L^2}{rt} \sqrt{1 - \mu^2}. \quad (13-65)$$

The minimum value of Eq. (13-63) is obtained when $m = 1$. Equation (13-63) then becomes

$$K = \frac{(1 + \beta^2)^2}{\beta^2} + \frac{12Z^2}{\pi^4 \beta^2 (1 + \beta^2)^2}. \quad (13-66)$$

Short Cylinders

For short cylinders, the curvature parameter Z can be set to zero and Eq. (13-66) becomes

$$K = (\beta + 1/\beta)^2. \quad (13-67)$$

Intermediate-Length Cylinders

As the ratio L/r increases, the quantity $\beta + 1$ can be approximated by β and Eq. (13-66) becomes

$$K = \beta^2 + \frac{12Z^2}{\pi^4\beta^6}. \quad (13-68)$$

Minimizing this quantity with respect to β gives

$$K = 1.038\sqrt{Z}. \quad (13-69)$$

Long Cylinders

For long cylinders, the buckling mode is similar to that of a circular ring and is elliptic in shape. Hence, $n = 2$ and

$$\beta = \frac{2L}{\pi r}$$

and Eq. (13-68) reduces to

$$K = \beta^2. \quad (13-70)$$

A plot of K defined by Eqs. (13-67), (13-69), and (13-70) is shown in Fig. 13-14.

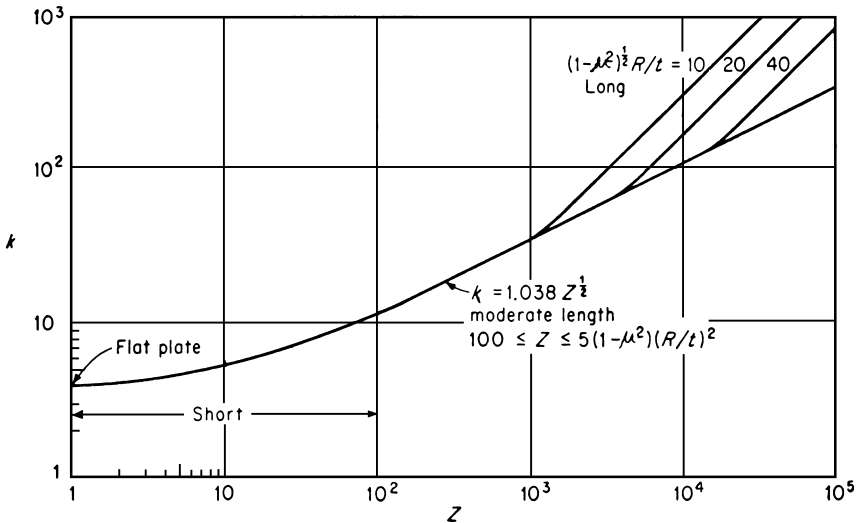


Figure 13-14. A plot of Z vs. K in Eq. (13-63). (Gerard 1962.)

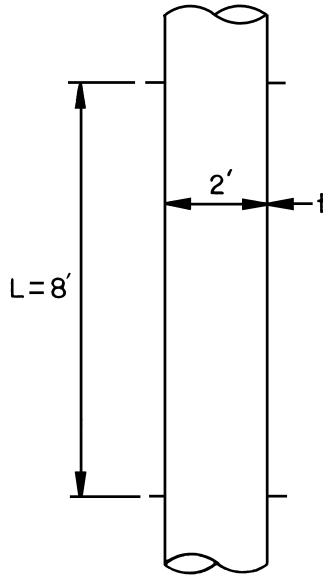


Figure 13-15. Stiffened pipe.

Example 13-4

The aluminum pipe shown in Fig. 13-15 is subjected to a vacuum pressure of 10 psi. The pipe has stiffeners spaced at 8-ft intervals. Let $E = 10,000$ ksi, factor of safety ($F.S.$) = 3.0, and $\mu = 0.25$. (a) Find the required thickness based on Eq. (13-62). (b) Compare the result with Sturm's Eq. (13-51).

(a)

$$\text{Let } t = 3/16 \text{ inch.}$$

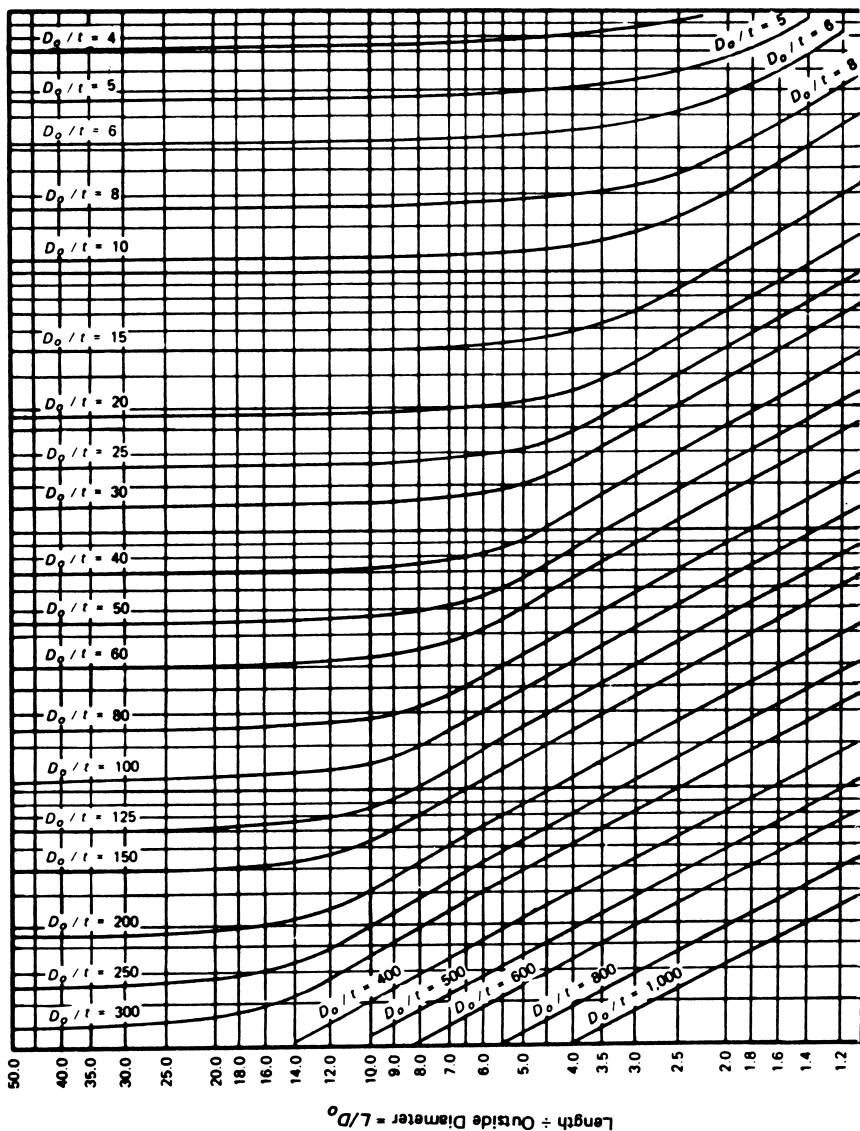
Then actual stress is

$$\begin{aligned}\sigma &= pr/t \\ &= 10 \times 12/0.1875 = 640 \text{ psi.}\end{aligned}$$

From Eq. (13-65)

$$\begin{aligned}Z &= \frac{L^2}{rt} \sqrt{1 - \mu^2} = \frac{96^2}{12 \times 0.1875} \sqrt{1 - 0.25^2} \\ &= 3966.\end{aligned}$$

From Eq. (13-69), $K = 1.038 \sqrt{Z} = 65.4$.



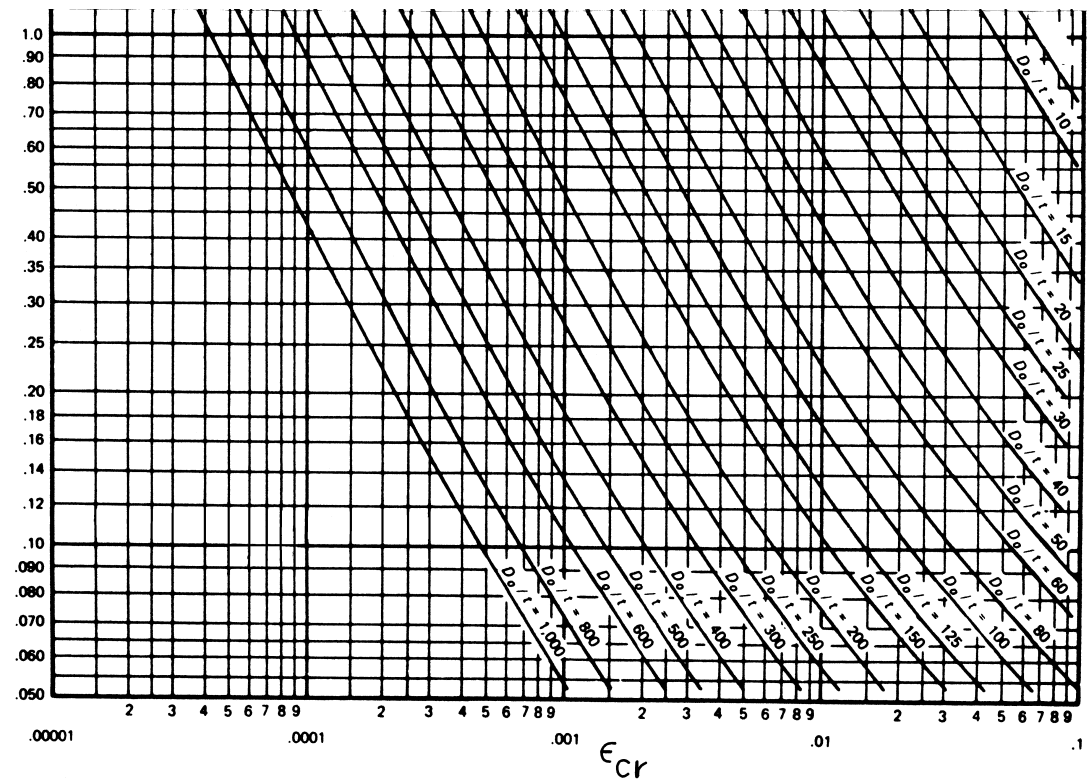


Figure 13-16. A plot of Eq. (13-71). (Courtesy of ASME.)

The allowable stress from Eq. (13-62) with a factor of safety of 3.0 is

$$\begin{aligned}\sigma_{\text{all}} &= \frac{\pi^2 \times 65.4 \times 10,000,000}{3 \times 12(1 - 0.25^2)} (0.1875/96)^2 \\ &= 730 \text{ psi} > 640 \text{ psi. ok.}\end{aligned}$$

(b)

$$L/r = 96/12 = 8, \quad D/t = 128.$$

From Fig. 13-9, $K = 7.2$.

From Eq. (13-51) with a factor of safety of 3,

$$\begin{aligned}p &= \frac{1}{3} \frac{7.2}{8} (10,000,000) (0.1875/12)^3 \\ &= 11.4 \text{ psi} \\ \sigma &= pr/t = 11.4 \times 12/0.1875 = 732 \text{ psi.}\end{aligned}$$

Problems

13-9 Solve Problem 13-4 by using Eq. (13-62). Is there any difference in the results? If so, why?

13-10 Solve Problem 13-5 by using Eq. (13-62).

13-6 Design Equations

The equations derived in Sections 13-1 through 13-5 are used in numerous codes and standards for design purposes. Some of these design equations are given in this section.

External Pressure

The ASME Power Boiler, Pressure Vessel, and Nuclear Reactor Codes use Eq. (13-51) as a basis for establishing design rules for external loads. This method permits the use of stress-strain curves of actual materials of construction to obtain allowable external pressure. This procedure prevents the possibility of calculating an allowable external pressure that results in a stress value that is above the yield stress of the material. Equation (13-51) for lateral and end pressure can be written as

$$\sigma_{cr} = p_{cr} r/t = \frac{K'}{2} E (t/2r)^2. \quad (13-71)$$

Define

$$\varepsilon_{cr} = \sigma_{cr}/E.$$

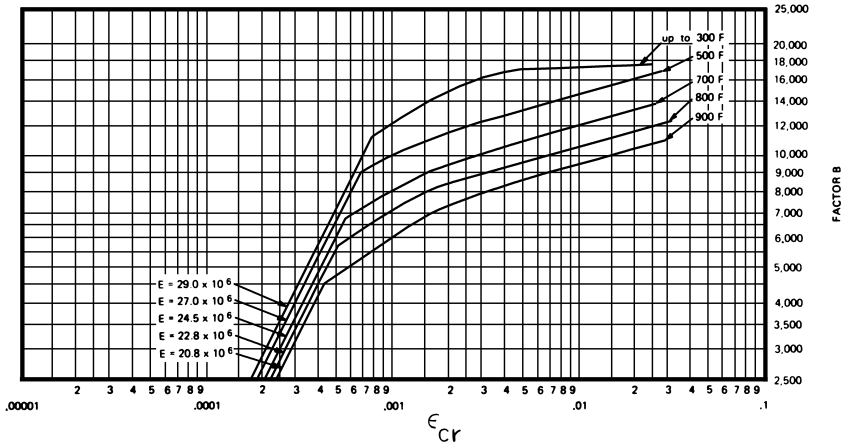


Figure 13-17. Chart for determining thickness of cylindrical and spherical shells under external pressure when constructed of carbon or low alloy steels and type 405 and type 410 stainless steels. (Courtesy of ASME.)

Equation (13-71) becomes

$$\epsilon_{cr} = \frac{K'}{2} (t/2r)^2. \quad (13-72)$$

A plot of this equation as a function of ϵ_{cr} , $L/2r$, and $t/2r$ is shown in Fig. 13-16. This figure is normally used by entering the values of $L/2r$ and $t/2r$ for a given shell and determining the critical strain ϵ_{cr} .

In order to determine the critical stress, plots of stress–strain diagrams are made for different materials at various temperatures. These diagrams are plotted by ASME on a log–log scale with the ordinate plotted as $\sigma_{cr}/2$ and abscissa as ϵ_{cr} . The value of $\sigma_{cr}/2$ in these diagrams is referred to by ASME as B . A sample of a stress–strain diagram for carbon steel is shown in Fig. 13-17. The ASME procedure consists of determining critical strain ϵ_{cr} from Fig. 11-16 and the B value from Fig. 13-17. The allowable external pressure is then calculated from

$$p = \frac{(2B)t}{r(F.S.)}. \quad (13-73)$$

If the value of ϵ_{cr} falls to the left of the material curves in Fig. 13-17, then the allowable external pressure is given by

$$p = \frac{\epsilon_{cr} E}{(r/t)(F.S.)}. \quad (13-74)$$

The ASME Codes use a factor of safety ($F.S.$) of 3.0 in Eqs. (13-73) and (13-74) for external pressure design.

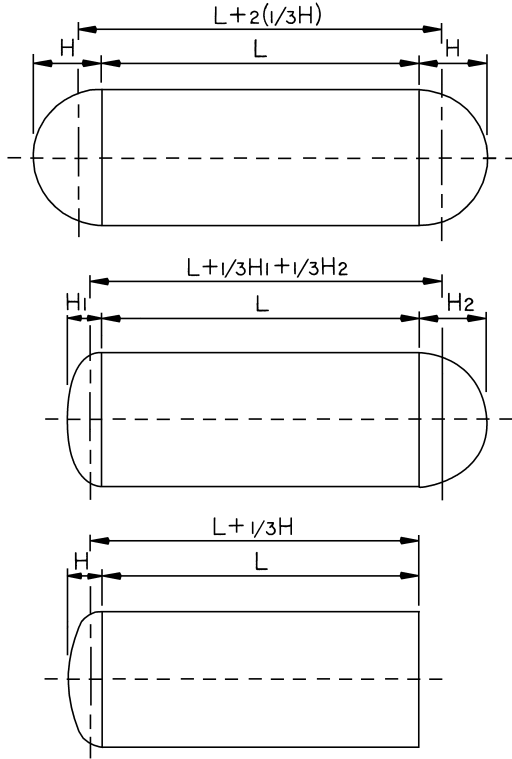


Figure 13-18. Cylindrical shells with various end attachments.

The ability of a cylindrical shell to resist external pressure increases with a reduction in its effective length. The ASME code gives rules for adding stiffening rings to reduce the effective length. Rules for the design of stiffening rings (Jawad and Farr 1989) are also given in the ASME code.

An empirical equation developed by the US Navy (Raetz 1957) for the buckling of cylindrical shells under lateral and axial pressure in the elastic range is given by

$$p_{cr} = \frac{2.42E}{(1 - \mu^2)^{3/4}} \frac{(t/2r)^{2.5}}{[L/2r - 0.45(t/2r)^{1/2}]}. \quad (13-75)$$

This equation is a good approximation of Eq. (13-51).

Tests that led to the development of Eq. (13-75) also showed that the effective cylindrical length for cylinders with end closures in the form of hemispherical or elliptical shape is equal to the length of the actual cylinder plus one-third the depth of the end closures as illustrated in Fig. 13-18.

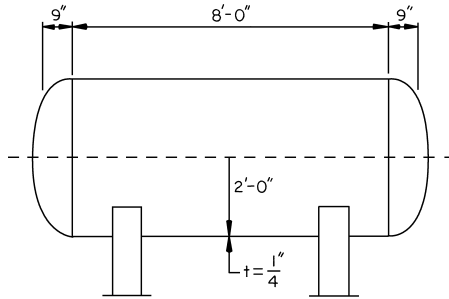


Figure 13-19. Storage vessel.

For structures with large r/t ratios, Eq. (13-75) can be simplified to

$$p_{cr} = \frac{2.42E}{(1 - \mu^2)^{3/4}} \frac{(t/2r)^{2.5}}{L/2r}.$$

Substituting into this equation the value of $\mu = 0.30$ for metallic structures and $\mu = 0.15$ for concrete structures results in

$$\left. \begin{aligned} p_{cr} &= \frac{0.92E(t/r)^{2.5}}{L/r} \text{ for metallic structures} \\ p_{cr} &= \frac{0.87E(t/r)^{2.5}}{L/r} \text{ for concrete structures} \end{aligned} \right] \quad (13-76)$$

Example 13-5

Determine the allowable pressure for the cylinder in Fig. 13-19 based on Eq. (13-75). Compare the result with that obtained from Eq. (13-51). Let $E = 16,000$ ksi, $\mu = 0.3$ and factor of safety ($F.S.$) = 4.

Solution

$$L/r = \frac{96 + 2 \times 9/3}{24} = 4.25$$

$$2r/t = 192.$$

From Eq. (13-75),

$$\begin{aligned} p &= (1/4) \left[\frac{2.42 \times 16,000,000}{(0.91)^{0.75}} \frac{(0.25/48)^{2.5}}{2.125 - 0.45(0.25/48)^{1.2}} \right] \\ &= 9.74 \text{ psi.} \end{aligned}$$

From Fig. 13-9, $K' = 18$. And from Eq. (13-51),

$$\begin{aligned} p &= \frac{1}{4 \times 8} \times 18 \times 16,000,000 (0.25/24)^3 \\ &= 10.17 \text{ psi.} \end{aligned}$$

Axial Compression

The allowable axial compressive stress for cylindrical shells in the ASME Code is obtained by using a factor of safety of 10 in Eq. (13-56) to account for the reduction in strength as shown in Fig. 13-12. This gives the following approximate equation for the elastic buckling:

$$\sigma = \frac{0.0625E}{r/t} \quad (13-77)$$

or in terms of strain

$$\varepsilon = \frac{0.0625}{r/t}.$$

In the inelastic region, the material charts (Fig. 13-17) must be used. Accordingly, the above strain equation must be multiplied by a factor of 2 to compensate for the fact that the ASME material charts are plotted with $B = \sigma/2$ which is half of the critical stress. Thus, we enter the charts with

$$\varepsilon = \frac{0.125}{r/t} \quad (13-78)$$

and read a value of B from the charts. This B value is the allowable compressive stress in the cylinder with a theoretical factor of safety of 10.

An empirical equation that is often used in the design of stacks and other self-supporting cylindrical structures made of low carbon steel was developed by the Chicago Bridge and Iron Company (Roark and Young 1975) as

$$\text{Allowable stress} = (X)(Y) \quad (13-79)$$

where

$$X = [1,000,000 t/r] \left[2 - \frac{2}{3} (100 t/r) \right] \quad \text{for } t/r < 0.015$$

$$X = 15,000 \text{ psi} \quad \text{for } t/r > 0.015$$

$$Y = 1 \quad \text{for } 2L/r \leq 60$$

$$Y = \frac{21,600}{18,000 + 2(L/r)} \quad \text{for } 2L/r > 60$$

Minimum $t = 1/4$ inch.

Another expression that is often used in determining the allowable compressive stress of steel cylindrical shells of flat-bottom oil storage tanks was developed by the API (API Standard 620). From Eq. (13-56) with $E = 30,000,000$ psi and a factor of safety of 10, the allowable compressive stress becomes

$$\sigma = 1,800,000 (t/r). \quad (13-80)$$

This allowable compressive stress is limited to 15,000 psi. A stress transition equation between this limit and Eq. (13-56) was developed by API as shown in

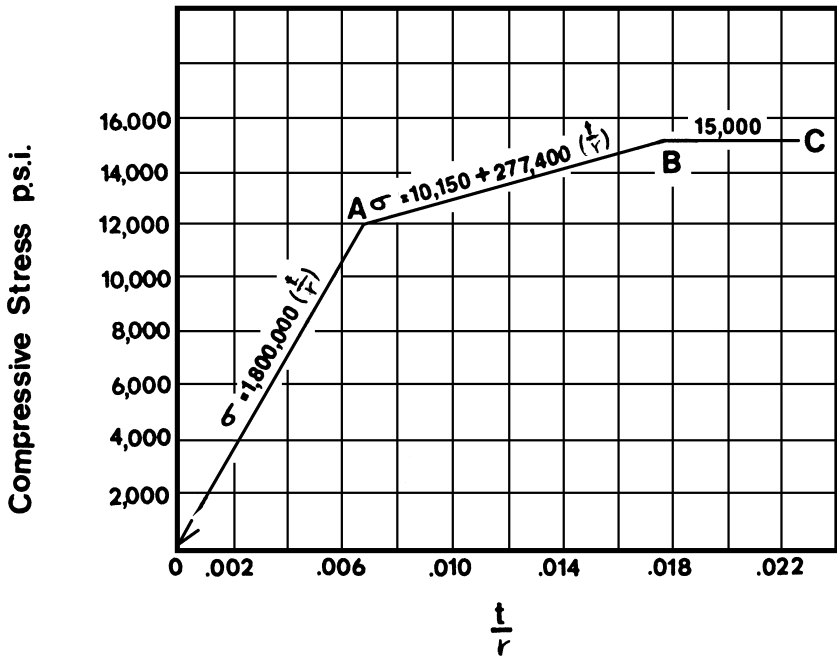


Figure 13-20. Allowable compressive stress in accordance with API standard 620. (Courtesy of American Petroleum Institute.)

Fig. 13-20. The API standard also includes curves for allowable stress of cylindrical shells subjected to biaxial stress combinations.

It should be noted that Eqs. (13-77) and (13-80) do not include any terms for the length of the cylinder. Hence, for extremely large L/r ratios, Euler's equation for column buckling may control the allowable stress rather than shell buckling and should be checked. In this case, the expression to be considered is

$$\sigma_{cr} = \frac{\pi^2 EI}{(KL)^2 A} \quad (13-81)$$

where

- A = material cross sectional area;
- I = material moment of inertia;
- K = 1.0 for cylinders with simply supported ends
- = 0.5 for cylinders with fixed ends
- = 2.0 for cantilever cylinders.

The critical axial compressive stress in reinforced concrete structures is obtained from Eqs. (13-56) and (13-57). Buckling stress for other loading conditions such

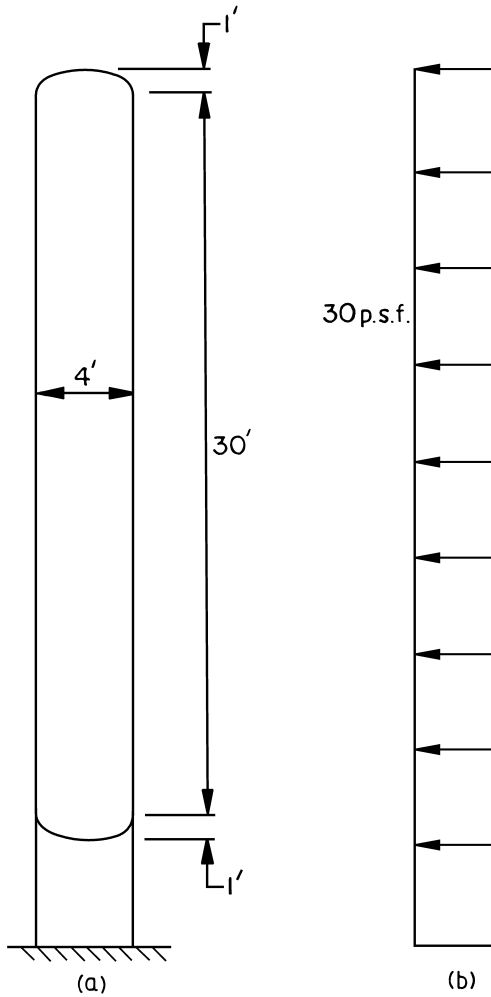


Figure 13-21. Reactor vessel.

as torsion and shear as well as various loading combinations are given in various publications such as that from NACA (Gerard and Becker 1957).

Example 13-6

The reactor shown in Fig. 13-21a is constructed of carbon steel with yield stress of 38 ksi and $E = 29,000$ ksi. The design temperature is 100°F . (a) Determine by

the ASME method the required thickness due to vacuum. (b) Check the thickness due to the wind loading shown in Fig. 13-21b.

Solution

(a) Vacuum condition

Try $t = 3/8$ inch.

$$L = 30 + (2)(1)/3 = 30.67 \text{ ft} = 368 \text{ inch.}$$

$$L/2r = 7.7 \quad \text{and} \quad 2r/t = 128.$$

From Fig. 13-16, $\varepsilon_{cr} = 0.00012$ inch/inch

From Fig. 13-17, ε_{cr} falls to the left of the material curve.

Hence, from Eq. (13-74)

$$p = \frac{(0.00012)(29,000,000)(0.375)}{3 \times 24} = 18.1 \text{ psi.} \quad \text{ok}$$

(b) The bending moment at the bottom of the cylinder due to wind load is given by

$$M = (30)(4)(31)^2/2 = 57,660 \text{ ft-lbs}$$

$$\text{stress } \sigma = \frac{M}{\pi r^2 t} = \frac{57,660 \times 12}{\pi (24)^2 (0.375)} = 1020 \text{ psi.}$$

From Eq. (13-78),

$$\varepsilon = \frac{0.125}{24/0.375} = 0.002 \text{ inch/inch}$$

and from Fig. 13-17, the allowable compressive stress is determined as

$$B = 15,000 \text{ psi} > 1020 \text{ psi.} \quad \text{ok.}$$

Check Euler's buckling ($K = 2.0$) by calculating

$$A = 2\pi r t = 56.55 \text{ in.}^2 \quad \text{and} \quad I = \pi r^3 t = 16,286 \text{ in.}^4$$

$$\sigma_{cr} = \frac{\pi^2 \times 29,000,000 \times 16,286}{(2 \times 360)^2 (56.55)}$$

$$\sigma_{cr} = 159,000 \text{ psi.}$$

Use 38,000 psi yield stress.

$$F.S. = 38,000/1020 = 37.$$

Hence, Euler's buckling does not control.

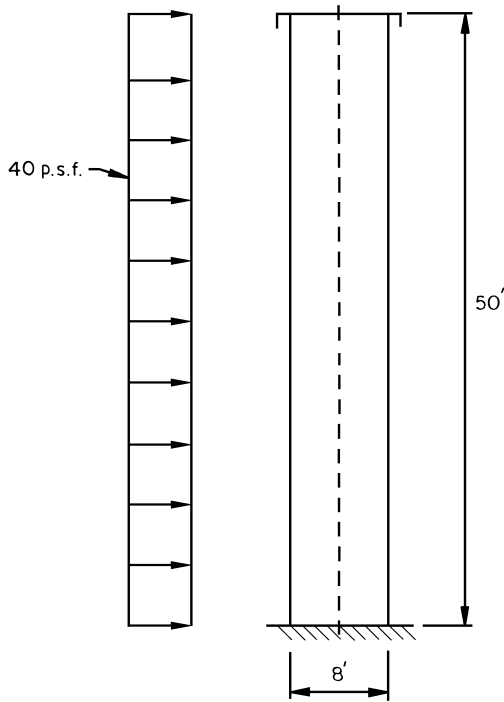


Figure 13-22. Low-carbon steel stack.

Example 13-7

The stack shown in Fig. 13-22 is subjected to an effective wind pressure of 40 psf. What is the required thickness? Use Eq. (13-79).

Solution

Maximum bending moment at bottom

$$M = (40)(8)(50)^2/2 = 400,000 \text{ ft-lbs}$$

Try $t = 1/4$ inch (minimum allowed by equation)

$$\begin{aligned}\sigma &= Mc/I = \frac{M}{\pi t r^2} = \frac{400,000}{\pi \times 0.25 \times 48^2} \\ &= 2660 \text{ psi.}\end{aligned}$$

From Eq. (13-79),

$$2(L/r) = 25$$

Hence,

$$Y = 1.0$$

$$t/r = 0.0052.$$

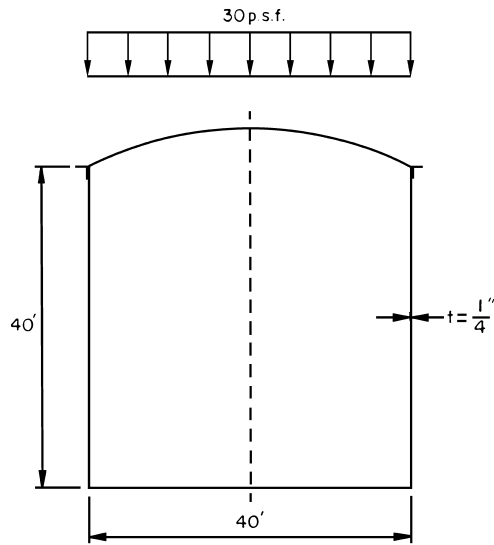
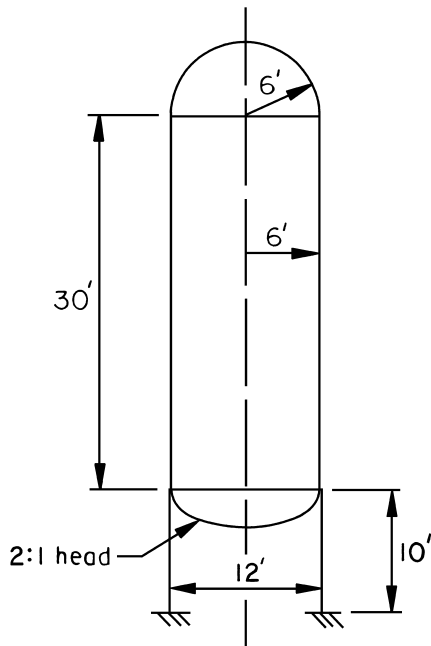


Figure 13-23. Flat bottom gasoline tank.



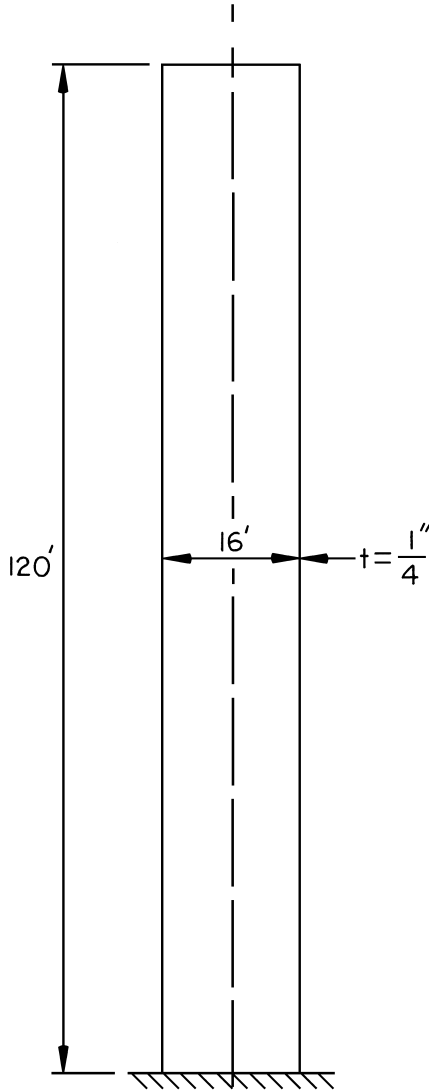
Prob. 13-11. Process reactor.

Also,

$$X = [1,000,000 \times 0.25/48][2 - (0.667)(100 \times 0.25/48)]$$

$$X = 8600 \text{ psi.}$$

$$\text{Allowable stress} = XY = 8600 \times 1.0 = 8600 \text{ psi} > 2660 \text{ psi.}$$



Prob. 13-14. Steel stack.

Example 13-8

The flat-bottom gasoline tank shown in Fig. 13-23 is subjected to a snow load of 30 psf. Calculate the actual stress in the cylinder due to snow load and compare it to the allowable stress.

Solution

Actual stress at bottom of cylindrical shell

$$\begin{aligned}\sigma &= \frac{p(\pi r^2)}{2\pi r t} \\ &= \frac{(30/144)(240)}{2(0.25)} = 100 \text{ psi.}\end{aligned}$$

From Fig. 13-20 with $t/r = 0.00104$,

$$\sigma = 1,800,000(0.00104) = 1870 \text{ psi} > 100 \text{ psi.}$$

Problems

13-11 Find the required thickness of the cylinder shown due to a full vacuum condition. Use Eq. (13-73) or (13-74).

13-12 Solve Problem 13-11 using Eq. (13-75).

13-13 What is the required thickness of the supporting cylinder shown in Prob. 13-11? The weight of the contents is 100,000 lbs.

13-14 What is the required thickness of the steel stack due to an effective wind load of 50 psf? Add the weight of the steel in the calculations. The thickness of the top 25 ft may be made different than the thickness of the bottom 25 ft. Weight of steel is 490 pcf. Use Eq. (13-79).

13-15 Calculate the required cylindrical shell thickness of the flat-bottom tank shown in Fig. 13-23 due to a snow load of 30 psf plus the dead weight of the roof and cylinder. Use Eq. (13-80).

Buckling of Shells of Revolution

14-1 Buckling of Spherical Shells

The buckling of spherical shells under external pressure has been investigated by numerous researchers. Von Karman (Von Karman and Tsien 1939) developed a solution that fits experimental data very closely. Taking a buckled section of a spherical shell, Fig. 14-1, he made the following assumptions:

1. The deflected shape is rotationally symmetric
2. The buckled length is small
3. The deflection of any element of the shell is parallel to axis of rotational symmetry
4. The effect of lateral contraction due to Poisson's ratio is neglected

Based on these assumptions and Fig. 14-1, the strain due to extension of the element is

$$\epsilon = \frac{\frac{dr}{\cos \theta} - \frac{dr}{\cos \alpha}}{dr / \cos \alpha}$$

$$\epsilon = \frac{\cos \alpha}{\cos \theta} - 1.$$

The strain energy of the extension of the element is

$$U_1 = \frac{ER^3}{2} (t/R) 2\pi \int_0^\beta \left(\frac{\cos \alpha}{\cos \theta} - 1 \right)^2 \sin \alpha \, d\alpha. \quad (14-1)$$

The strain energy due to compression of the shell prior to buckling is

$$U_2 = \frac{-ER^3}{2} (t/R) 2\pi \int_0^\beta \left(\frac{pR}{2Et} \right)^2 \alpha \, d\alpha. \quad (14-2)$$

Similarly, the strain energy due to bending is given by

$$U_3 = \frac{ER^3}{2} (t/R) \frac{3\pi}{6} \int_0^\beta \sin \alpha \left[\left(\frac{\cos \theta}{\cos \alpha} \frac{d\theta}{d\alpha} - 1 \right)^2 + \left(\frac{\sin \theta}{\sin \alpha} - 1 \right)^2 \right] d\alpha. \quad (14-3)$$

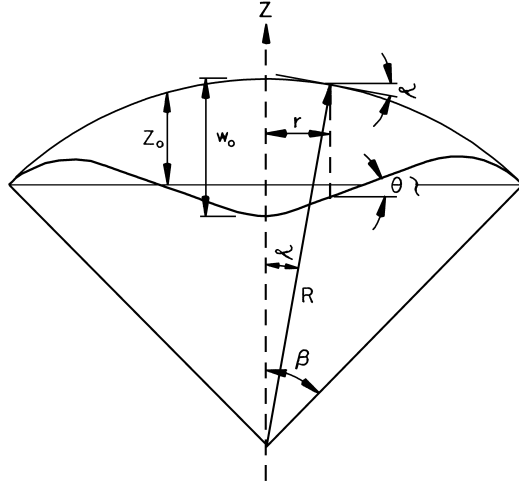


Figure 14-1. Buckled shape of a spherical shell.

The external work is equal to the applied pressure times the volume included between the deflected and original surfaces. This can be expressed as

$$W = R^3 \pi \int_0^\beta \sin^2 \alpha (\tan \theta - \tan \alpha) \cos \alpha \, d\alpha. \quad (14-4)$$

The total potential energy, Π , of the system is given by

$$\Pi = U_1 + U_2 + U_3 - W. \quad (14-5)$$

The terms in Eq. (14-5) can be simplified substantially by assuming β to be small. Then, expanding the sine and cosine expressions into a power series and neglecting terms higher than the third order, Eq. (14-5) becomes

$$\begin{aligned} \Pi = E(t/R) \int_0^\beta \left[\frac{1}{2} (\theta^2 - \alpha^2) - \frac{PR}{2Et} \right]^2 \alpha \, d\alpha \\ + \frac{E(t/R)^3}{12} \int_0^\beta \left[\left(\frac{d\theta}{d\alpha} - 1 \right)^2 + \left(\frac{\theta}{\alpha} - 1 \right)^2 \right] \alpha \, d\alpha + p \int_0^\beta \alpha^2 (\theta - \alpha) \, d\alpha. \end{aligned} \quad (14-6)$$

Minimizing this equation with respect to θ results in an equation that is too cumbersome to solve. Accordingly, Von Karman assumed an expression for θ that satisfies the boundary condition $\theta = 0$ at $\alpha = 0$ and $\theta = \beta$ at $\alpha = \beta$, in the form

$$\theta = \left\{ 1 - K \left[1 - (\alpha/\beta)^2 \right] \right\} \beta \quad (14-7)$$

where K is a constant.

Substituting Eq. (14-7) into the expression

$$\frac{\partial \Pi}{\partial K} = 0$$

and utilizing the relationship $\sigma = pR/2t$ we get

$$\frac{\sigma}{E} = \frac{1}{70} \beta^2 (28 - 21K + 4K^2) + \frac{4}{3} (t/R)^2 \frac{1}{\beta^2}. \quad (14-8)$$

The ordinate Z_o in Fig. 14-1 is calculated from

$$Z_o + \int_0^\beta (dz/dr) dr = 0$$

or

$$Z_o = R \int_0^\beta \tan \theta \cos \alpha \, d\alpha.$$

The ordinate of the shell at the center before deflection is given by $R(1 - \cos \beta)$. The deflection, w_o , at the center is then expressed by

$$w_o = R \int_0^\beta (\tan \alpha - \tan \theta) \cos \alpha \, d\alpha$$

or, assuming β to be small,

$$w_o = R \int_0^\beta (\alpha - \theta) \, d\alpha. \quad (14-9)$$

Substituting Eq. (14-9) into Eq. (14-7) gives

$$K = \frac{4w_o}{R\beta^2} \quad (14-10)$$

Substituting Eq. (14-10) into Eq. (14-8) results in

$$\frac{\sigma}{E} = \frac{2}{5} \beta^2 - \frac{6}{5} \frac{w_o}{R} + \left[\frac{32}{35} \frac{w_o^2}{R^2} + \frac{4}{3} \frac{t^2}{R^2} \right] \frac{1}{\beta^2}. \quad (14-11)$$

The minimum value of Eq. (14-11) is obtained by differentiating with respect to β^2 and equating to zero. This gives

$$\beta^2 = \sqrt{\frac{16}{7} (w_o/R)^2 + \frac{10}{3} (t/R)^2}$$

and Eq. (14-11) becomes

$$\frac{\sigma R}{Et} = \frac{4}{5} \left[\sqrt{\frac{16}{7} (w_o/t)^2 + \frac{10}{3}} - \frac{3}{2} (w_o/t) \right]. \quad (14-12)$$

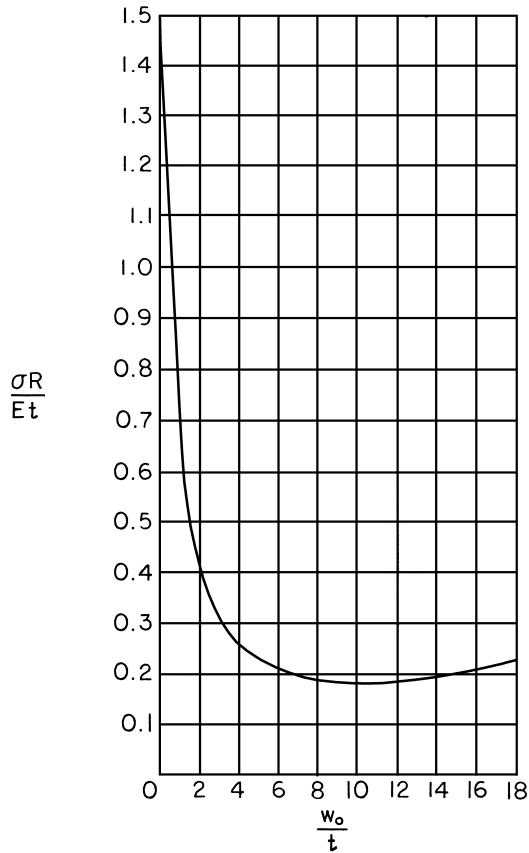


Figure 14-2. A plot of Eq. (14-12).

A plot of this equation is shown in Fig. 14-2. The minimum buckling value is

$$\left. \begin{aligned} \sigma_{cr} &= 0.183Et/R \\ P_{cr} &= 0.366E \frac{t^2}{R^2} \end{aligned} \right\} \quad (14-13)$$

Example 14-1

A hemispherical shell with a radius of 72 inches is subjected to full vacuum. Determine the required thickness if a factor of safety (*F.S.*) of 4 is used for buckling. Let $E = 30,000$ ksi

Solution

From Eq. (14-13), with $F.S. = 4$, we get

$$\begin{aligned} t &= R \sqrt{\frac{4p}{0.366E}} \\ &= 72 \sqrt{\frac{4 \times 14.7}{0.366 \times 30,000,000}} \\ &= 0.17 \text{ inch.} \end{aligned}$$

Problems

14-1 Find the required thickness of the hemispherical shell in Problem 13-4. Use a factor of safety of 3.0.

14-2 A spherical aluminum diving chamber is under 1000 ft of water. Determine the required thickness. Let $R = 36$ inches, factor of safety ($F.S.$) = 1.5, and $E = 10,000$ ksi.

14-2 Buckling of Stiffened Spherical Shells

Equation (14-13) can be written in terms of buckling pressure as

$$p_{cr} = 0.366E(t/R)^2 \quad (14-14)$$

or

$$t = R \sqrt{\frac{p_{cr}}{0.366E}}. \quad (14-15)$$

Equation (14-15) shows that for a given external pressure and modulus of elasticity, the required thickness is proportional to the radius of the spherical section. As the radius gets larger, so does the thickness. One procedure for reducing the membrane thickness is by utilizing stiffened shells. Thus, for the shell shown in Fig. 14-13 with closely spaced stiffeners, the buckling pressure (Buchert 1964) is obtained by modifying Eq. (14-14) as

$$p_{cr} = 0.366E \left(\frac{t_m}{R} \right)^2 \left(\frac{t_b}{t_m} \right)^{3/2} \quad (14-16)$$

in which

t_m = effective membrane thickness
 $= t + (A/d)$

t_b = effective bending thickness
 $= \left(\frac{12I}{d} \right)^{1/3}$

where

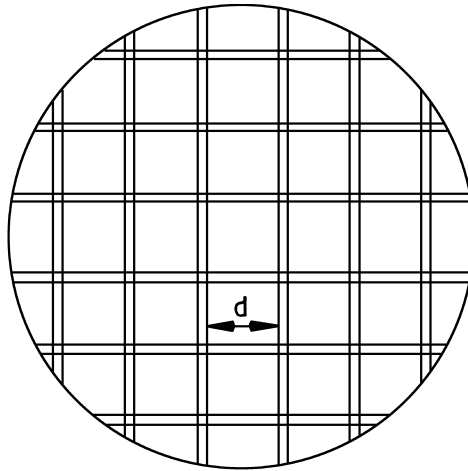
A = area of stiffening ring;

d = spacing between stiffeners;

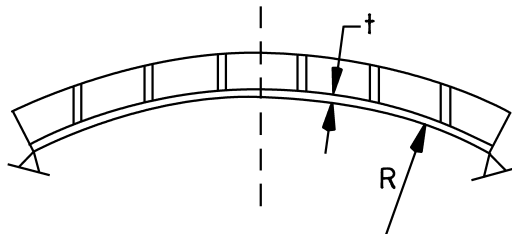
I = moment of inertia of ring;

t = thickness of shell.

For large spherical structures such as large tank roofs and stadium domes, the stiffener spacing in Fig. 14-3 increases significantly. In this case, the



(a)



(b)

Figure 14-3. Stiffened dome.

composit buckling strength of shell and stiffeners (Buchert 1966) is expressed by

$$p_{cr} = 0.366E \left(\frac{t}{R} \right)^2 \left(1 + \frac{12I}{dt^3} \right)^{1/2} \left(1 + \frac{A}{dt} \right)^{1/2} \quad (14-17)$$

Local buckling of the shell between the stiffeners must also be considered for large-diameter shells. One such equation is given by

$$p_{cr} = 7.42 \frac{Et^3}{Rd^2}. \quad (14-18)$$

It should be noted that large edge rotations and deflections can reduce the value obtained by Eq. (14-17) significantly.

Equation (14-17) is also based on the assumption that the spacing of the stiffeners is the same in the circumferential and meridional directions. Other equations can be developed (Buchert 1966) for unequal spacing of stiffeners.

Problems

14-3 Determine the allowable external pressure on the stiffened head in Fig. 14-3. Assume $R = 13$ ft, $t = 3/8$ inch, $d = 8$ inches, size of stiffeners is 4 inch $\times$ 3/8 inch, $E = 29,000$ ksi, and a factor of safety ($F.S.$) of 10.

14-4 Use Eqs. (14-17) and (14-18) to determine the required thickness of dome roof and the size and spacing of stiffeners. Let $R = 200$ ft, $E = 29,000$ ksi, $p = 85$ psf, and factor of safety = 10.

14-3 Buckling of Conical Shells

The derivation of the equations for the buckling of conical shells is fairly complicated and beyond the scope of this book. The derivation (Niordson 1947) for the buckling pressure of the cone shown in Fig. 14-4 consists of obtaining expressions for the work done by the applied pressure, membrane forces, stretching of the middle surface, and bending of the cone. The total work is then minimized to obtain a critical pressure expression in the form

$$p_{cr} = \frac{Et}{\rho_o a_o^2} f \frac{t^2 a_o^4}{(1 - \mu^2) \rho_o^2} \quad (14-19)$$

where

E = modulus of elasticity;

t = thickness of cone;

$a_o = \lambda \rho (1 - \beta/2)$;

$\lambda = \pi/\ell$;

$\rho_o = \rho (1 - \beta/2)$;

$\beta = \frac{\ell}{\rho} \tan \alpha$.

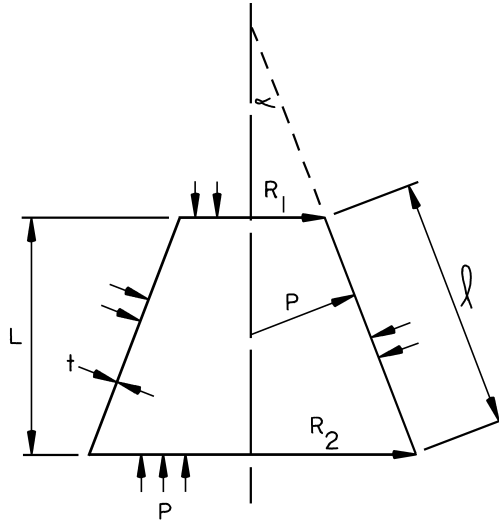


Figure 14-4. Conical section.

Equation (14-19) is very cumbersome to use due to the iterative process required. Seide showed (Seide 1962) that Eq. (14-19) for the buckling of a conical shell is similar to the equation for the buckling of a cylindrical shell having a length equal to the slant length of the cone and a radius equal to the average radius of the cone. He also showed that the buckling of a cone is affected by the function $f(1 - R_1/R_2)$ and is expressed as

$$p_{cr} = \bar{p}f(1 - R_1/R_2) \quad (14-20)$$

where

$\bar{p}$ = pressure of equivalent cylinder as defined above;
 f = cone function as defined in Fig. 14-5.

By various substitutions (Jawad 1980), it can be shown that Eq. (14-20) can be transferred to the form of Eq. (13-76) as

$$p_{cr} = \frac{0.92E(t_e/R_2)^{2.5}}{L_e/R_2} \text{ for metallic cones}$$

$$p_{cr} = \frac{0.87E(t_e/R_2)^{2.5}}{L_e/R_2} \text{ for concrete cones} \quad (14-21)$$

where

t_e = effective thickness of cone

$$= t \cos \alpha$$

t = thickness of cone

$$(14-22)$$

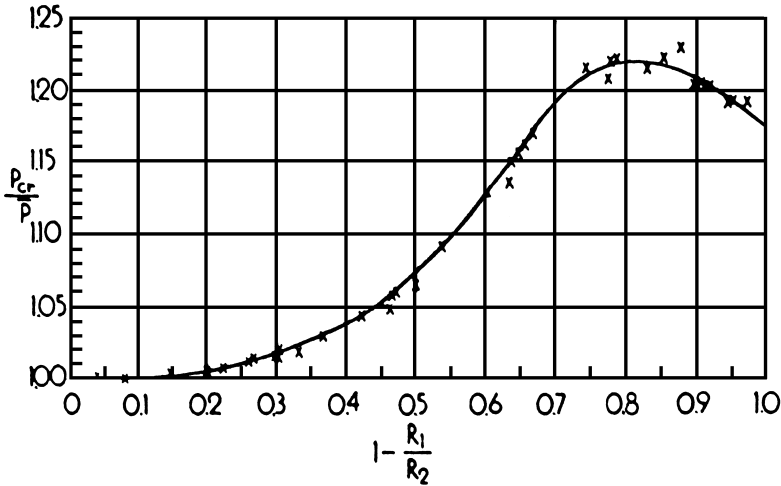


Figure 14-5. Plot of Eq. (14-20). (Jawad 1980)

and

L_e = effective length of cone

$$= \frac{L}{2} \left(1 + R_1/R_2 \right). \quad (14-23)$$

Thus, conical shells subjected to external pressure may be analyzed as cylindrical shells with an effective thickness and length as defined by Eqs. (14-22) and (14-23).

Example 14-2

Determine the allowable external pressure, at room temperature, for the steel cone shown in Fig. 14-6 by using (a) Eq. (14-21) and (b) Eq. (13-73). Let $F.S. = 3.0$, and $E = 30,000$ ksi.

Solution

(a)

From Fig. 14-6, $\alpha = 17.35^\circ$

From Eq. (14-22), $t_e = (3/32)(\cos 17.35) = 0.0895$ inch.

From Eq. (14-23), $L_e = \frac{32}{2} \left(1 + \frac{20}{30} \right) = 26.67$ inch.

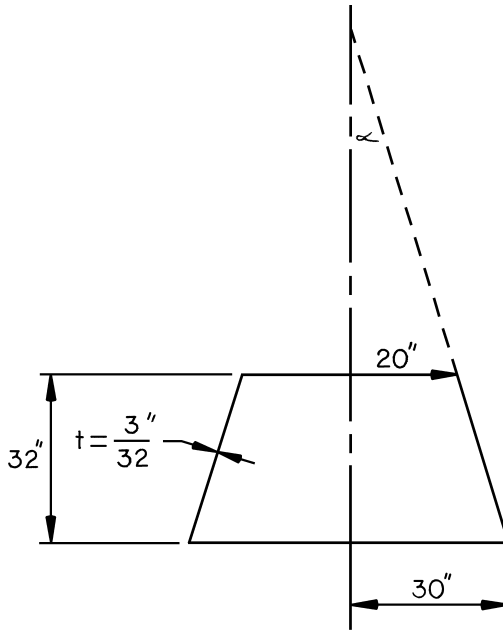


Figure 14-6. Steel conical section.

Hence, from Eq. (14-21),

$$p = \frac{0.92 \times 30,000,000 \left(\frac{0.0895}{30} \right)^{2.5}}{(26.67/30)(3)}$$

$$= 5.03 \text{ psi.}$$

(b)

$$L_e/D_2 = 26.67/60 = 0.44$$

$$D_2/t_e = 60/0.0895 = 670$$

From Fig. 13-16, $\epsilon = 0.00018 \text{ inch/inch}$

From Fig. 13-17, $B = 2500 \text{ psi}$

and from Eq. (13-73),

$$p = \frac{(2 \times 2500)(0.0895)}{(30)(3)}$$

$$= 4.97 \text{ psi.}$$

It should be remembered that there is an advantage in using method (b) in that the actual stress–strain diagram of the material is used. This takes into account the plastic region if σ falls in that range.

Problems

14-5 Determine the required thickness of the cone shown in Prob. 11-9 due to full vacuum. Use (a) Eq. (14-21) and (b) Eq. (13-73). Let $E = 16,000$ ksi and factor of safety ($F.S.$) = 2.50.

14-6 Determine the required thickness of the cone shown in Fig. 14-6 due to full vacuum. Let $E = 30,000$ ksi and use $F.S. = 3.0$.

14-4 Design Considerations

Spherical Shells

Extensive tests on spherical shells (Kollar and Dulacska 1984) have shown that the theoretical buckling Eq. (14-13) must be reduced further to a value of

$$\sigma_{cr} = 0.125Et/R \quad (14-24)$$

in order to match experimental data. This equation may also be written in terms of critical pressure as

$$p_{cr} = \frac{E}{4(R/t)^2}. \quad (14-25)$$

ASME uses Eq. (14-25) with a factor of safety of 4.0 to obtain the permissible external pressure on a spherical shell. Hence,

$$p = \frac{0.0625E}{(R/t)^2}. \quad (14-26)$$

Ellipsoidal Shells

An approximate equation for the buckling of ellipsoidal shells is similar to Eq. (14-13) for spherical shells and is given by

$$p_{cr} = 0.366E \frac{t^2}{r_1 r_2} \quad (14-27)$$

where r_1 and r_2 are defined in Fig. 8-8.

Shallow Heads

For shallow ellipsoidal shells under external pressure, the region near the knuckle area is in tension. The remaining surface can be approximated by a spherical shell and Eq. (14-27) is simplified by letting $r_1 = r_2$.

Conical Shells

Conical shells subjected to external pressure are designed as cylindrical shells with an effective thickness and length as given by Eqs. (14-22) and (14-23).

Hyperbolic Paraboloid Sheets

The critical buckling of hyperbolic paraboloid sheets of revolution with stiffening rings at top and bottom due to radial and axial loads is given by

$$p_{cr} = \lambda E(t/r)^2 \quad (14-28)$$

where, r is the smallest radius, Fig. 9-16, at the throat. The value of λ depends on many factors such as height of structure, thickness, r , r_1 , and r_2 . The value is tabulated in Kollar (Kollar and Dulacska 1984) for various parameters.

Other forms of buckling (Kollar and Dulacska 1984) such as free-edge, local, and axisymmetric buckling must also be investigated. Usually, these forms of buckling are less severe than that given by Eq. (12-28) except for specific conditions.

Various Shapes

Further theoretical and experimental research is still needed to establish buckling strength of various configurations and shapes. This includes eccentric cones, torispherical shells, toriconical shells, and stiffened shells. Additionally, the effect of out-of-roundness and edge conditions on the buckling strength of shells needs further investigation.

Vibration of Shells

15-1 Cylindrical Shells

The governing equations for the vibration of cylindrical shells are based, in part, on the equations derived in Chapter 10. These equations, which are based on symmetric loading, were derived from Fig. 10-2, which is reproduced here as Fig. 15-1a in a more general form. Additionally, non-symmetric loading such as torsional moments and shear strains, Fig. 15-1b, must be considered in the derivation of the equations for shell vibration. Also, deformations u , v , and w as well as inertial forces in the axial, circumferential and radial directions must be taken into consideration. The resulting differential equations in the axial, circumferential and radial directions are

$$r^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}(1 - \mu) \frac{\partial^2 u}{\partial \theta^2} + \frac{r}{2}(1 + \mu) \frac{\partial^2 v}{\partial x \partial \theta} - \mu r \frac{\partial w}{\partial x} = \frac{\rho}{E} (1 - \mu^2) r^2 \frac{\partial^2 u}{\partial t^2} \quad (15-1)$$

$$\frac{r}{2}(1 + \mu) \frac{\partial^2 u}{\partial x \partial \theta} + \frac{r^2}{2}(1 - \mu) \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial \theta^2} - \frac{\partial w}{\partial \theta} = \frac{\rho}{E} (1 - \mu^2) r^2 \frac{\partial^2 v}{\partial t^2} \quad (15-2)$$

$$\mu r \frac{\partial u}{\partial x} + \frac{\partial v}{\partial \theta} - w - k \left(r^4 \frac{\partial^4 w}{\partial x^4} + 2r^2 \frac{\partial^4 w}{\partial x^2 \partial \theta^2} + \frac{\partial^4 w}{\partial \theta^4} \right) = \frac{\rho}{E} (1 - \mu^2) r^2 \frac{\partial^2 w}{\partial t^2} \quad (15-3)$$

Where,

$$k = \frac{t^2}{12r^2} \quad (15-4)$$

and,

E = modulus of elasticity

r = radius of cylinder

t = thickness of cylinder

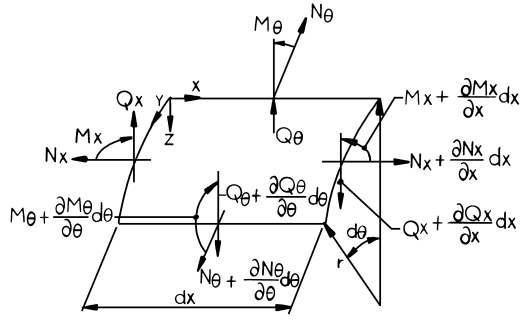
u = axial deflection along the x -axis

v = circumferential deformation along the θ -axis

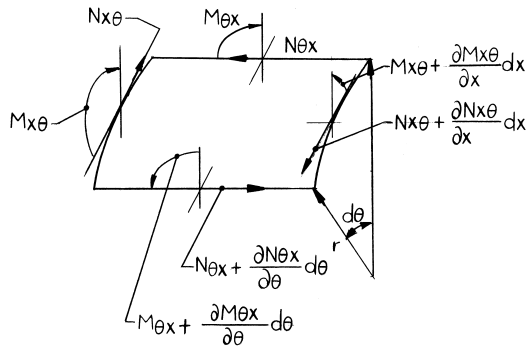
w = radial deformation

μ = Poisson's ratio

ρ = mass density per unit volume (density/acceleration)



(a)



(b)

Figure 15-1. Forces in a cylindrical segment.

Equations (15-1) through (15-3) may be written in matrix form as

$$[L_{DM}] \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{\rho}{E} (1 - \mu^2) r^2 \frac{\partial^2}{\partial t^2} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad (15-5)$$

Let the deflections during free vibration be represented by

$$u = C_1 \cos \eta x \cos n\theta \cos \omega t$$

$$v = C_2 \cos \eta x \cos n\theta \cos \omega t$$

$$w = C_3 \cos \eta x \cos n\theta \cos \omega t$$

For cylindrical shells with simply supported edges, the boundary conditions can be taken as $N_x = M_x = v = w = 0$ at $x = 0$ and at $x = L$. Substituting the above values for u , v , and w into Eqs. (15-1) through (15-3) and using the stated boundary conditions results, after rearranging terms, in three simultaneous equations that can be written in matrix form as

$$\begin{bmatrix} \Omega^2 - A_1 & \frac{n\lambda}{2}(1 + \mu) & -\mu\lambda \\ \frac{n\lambda}{2}(1 + \mu) & \Omega^2 - A_2 & n \\ -\mu\lambda & n & \Omega^2 - A_3 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (15-6)$$

where,

$$\Omega^2 = \frac{\rho}{E} (1 - \mu^2) r^2 \omega^2 \quad (15-7)$$

and,

$$\lambda = \eta r = \frac{m\pi r}{L}$$

$$A_1 = \lambda^2 + \frac{n^2}{2}(1 - \mu)$$

$$A_2 = \frac{\lambda^2}{2}(1 - \mu) + n^2$$

$$A_3 = 1 + k(\lambda^2 + n^2)^2$$

A non-trivial solution of Eq. (15-6) is obtained by setting the determinant of the coefficient matrix to zero. This yields a quadratic equation of the form

$$\Omega^6 - \Delta_1 \Omega^4 + \Delta_2 \Omega^2 - \Delta_3 = 0 \quad (15-8)$$

where,

$$\Delta_1 = 1 + \frac{1}{2}(3 - \mu)(\lambda^2 + n^2) + k(\lambda^2 + n^2)^2$$

$$\Delta_2 = \frac{1}{2}(1 - \mu) \left[(3 + 2\mu)\lambda^2 + n^2 + (\lambda^2 + n^2)^2 + \frac{(3 + \mu)}{(1 - \mu)} k(\lambda^2 + n^2)^3 \right]$$

$$\Delta_3 = \frac{1}{2}(1 - \mu) \left[(1 - \mu^2)\lambda^4 + k(\lambda^2 + n^2)^4 \right]$$

The solution of Eq. (15-8) results in three circumferential natural frequencies, n , for each assumed longitudinal frequency, m . The nodal patterns for the first few frequencies are shown in Fig. 15-2.

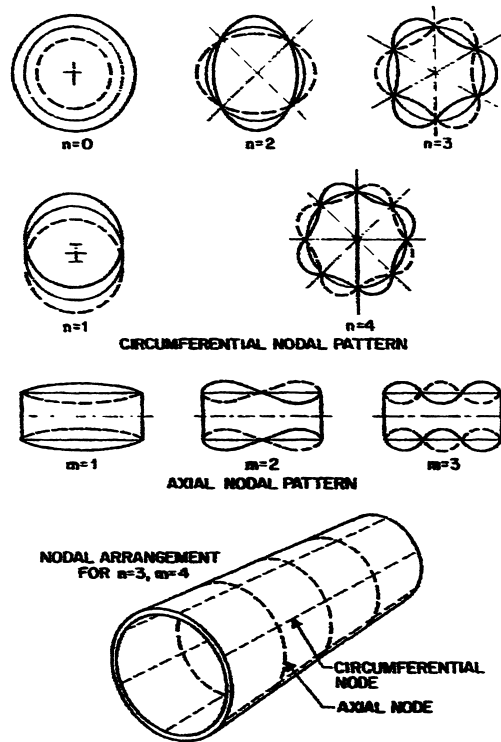


Figure 15-2. Nodal Patterns for Cylindrical Shells (Leissa, 1973).

Equations (15-1) through (15-3) are referred to as the Donnell-Mushtari equations (Markus 1988). Others have derived equations that modify the Donnell-Mushtari equations in order to improve the accuracy of the results. Most of these derivations can be presented as Donnell-Mushtari equations with some modifiers. Leissa (Leissa 1973) made a comprehensive comparison of the results of eleven different methods with results from the theory of elasticity. Leissa evaluated the methods of Donnell-Mushtari, Love-Timoshenko, Goldenveizer-Novozhilov, Biezeno-Grammel, Flugge, Reissner-Naghdi-Berry, Sanders, Vlasov, Epstein-Kennard, Houghton-Johns, and Kennard simplified. Leissa's results indicate that for $r/t = 20$ and (L/mr) between 1 and 100 the error of all methods is less than 1% for $n = 0, 1$, and 2. For $n = 3$ and 4, all methods have an error of less than 1% with the exception of the Donnell-Mushtari equations where the error ranges from 2.9% for $(L/mr) = 1$ to 12.87% for $(L/mr) = 100$. The error percentage decreases somewhat for shells with $r/t = 500$.

The fundamental natural frequency of a cylinder with a specific geometry, physical properties and boundary conditions is a function of m and n . This is

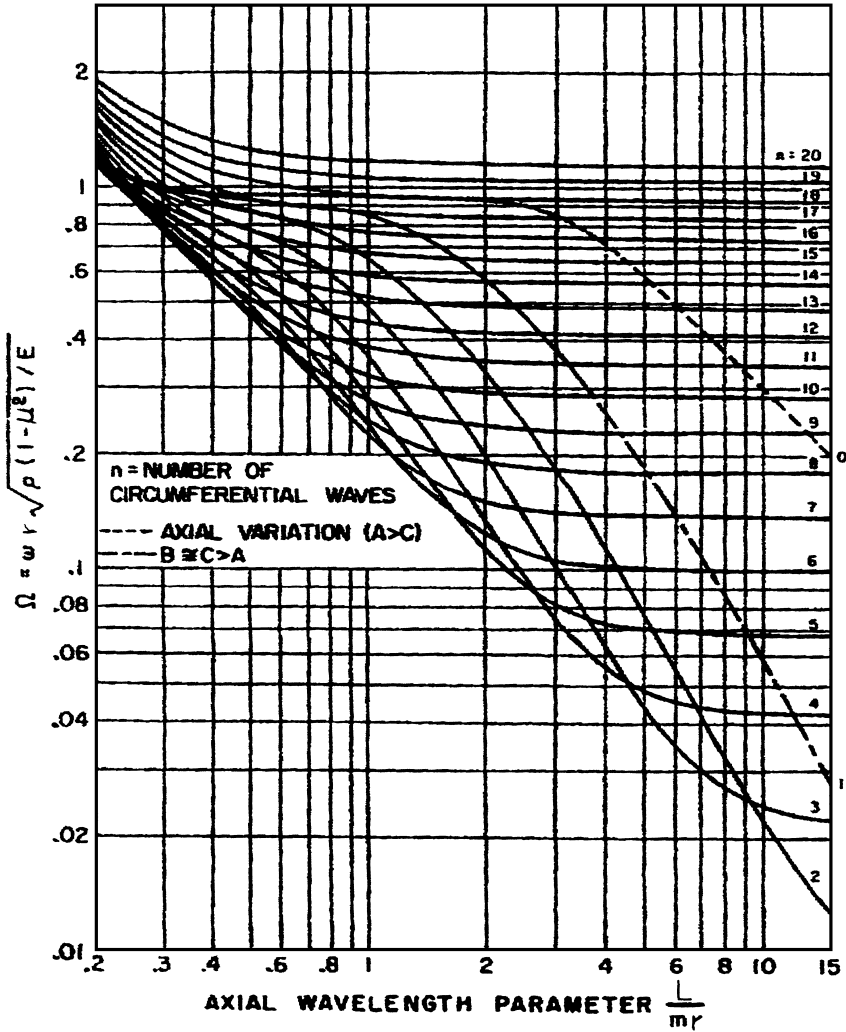


Figure 15-3. Fundamental Frequency parameter Ω as a Function of L/mr Using Flugge's Theory for $r/t = 100$ and $\mu = 0.30$. (Leissa, 1973).

illustrated in Fig. 15-3 using Flugge's theory for a cylinder with $r/t = 100$ and $\mu = 0.30$. The figure shows that the value of n corresponding to the lowest frequency is a function of the quantity L/mr . Thus it is necessary, when calculating the fundamental natural frequency of a cylinder without the aid of a chart, to try different values of n for any given value of m until a minimum value of the natural frequency is obtained.

Example 15-1

What is the natural frequency of a cylindrical shell simply supported at the edges if $r = 2$ inches, $L = 15$ inches, $E = 30,000$ ksi, $\mu = 0.3$, $t = 1/32$ inch, and $\rho = 0.733 \times 10^{-3}$ lb-sec²/in⁴?

Solution

The minimum value of m is taken as $m = 1$. Three values of n are tried in order to obtain the minimum natural frequency. They are $n = 2, 3$, and 4 . It was found that the minimum natural frequency is obtained when $n = 3$. The following calculations are based on $n = 3$.

$$\lambda = \pi(2)/15 = 0.41888$$

$$k = (1/12)(0.03125/2)^2 = 2.03451 \times 10^{-5}$$

$$\begin{aligned}\Delta_1 &= 1 + (0.5)(3 - 0.3)(0.41888^2 + 3^2) + (2.03451 \times 10^{-5})(0.41888^2 + 3^2)^2 \\ &= 13.38858\end{aligned}$$

$$\begin{aligned}\Delta_2 &= (0.5)(1 - 0.3) \left[(3 + 2 \times 0.3)(0.41888)^2 + 3^2 + (0.41888^2 + 3^2)^2 \right. \\ &\quad \left. + \frac{(3 + 0.3)}{(1 - 0.3)} (2.03451 \times 10^{-5})(0.41888^2 + 3^2)^3 \right] = 32.86318\end{aligned}$$

$$\begin{aligned}\Delta_3 &= (0.5)(1 - 0.3) \left[(1 - 0.3^2)(0.41888)^4 + (2.03451 \times 10^{-5})(0.41888^2 + 3^2)^4 \right] \\ &= 0.06028\end{aligned}$$

Equation (15-8) is written as

$$\Omega^6 - (13.38858)\Omega^4 + (32.86318)\Omega^2 - 0.06028 = 0$$

This cubic equation is of the form

$$y^3 + py^2 + qy + r = 0.$$

Define

$$a = (1/3)(3q - p^2)$$

$$b = (1/27)(2p^3 - 9pq + 27r)$$

$$\phi = \cos^{-1} \left(\frac{-b/2}{(-a^3/27)^{1/2}} \right)$$

then,

$$y_1 = \Omega_1^2 = 2(-a/3)^{1/2} \cos\left(\frac{\phi}{3}\right) - p/3$$

$$y_2 = \Omega_2^2 = 2(-a/3)^{1/2} \cos\left(\frac{\phi}{3} + 120\right) - p/3$$

$$y_3 = \Omega_3^2 = 2(-a/3)^{1/2} \cos\left(\frac{\phi}{3} + 240\right) - p/3$$

Thus, for $p = -13.38858$, $q = 32.86318$, and $r = -0.06028$ we get

$$a = (1/3)[3 \times 32.86318 - (-13.38858)^2] = -26.88818$$

$$b = (1/27)[2(-13.38858)^3 - 9(-13.38858)(32.86318) + 27(-0.06028)] = -31.17114$$

$$\phi = \cos^{-1}\left(\frac{-(-31.17114)/2}{[-(-26.88818)^3/27]^{1/2}}\right) = \cos^{-1}(0.58085) = 54.48979^\circ$$

$$2(-a/3)^{1/2} = 2[-(-26.88818)/3]^{1/2} = 5.98756$$

$$p/3 = -4.46295$$

$$\phi/3 = 18.16326^\circ$$

and

$$\Omega_1^2 = (5.98756) \cos(18.16326) - (-4.46295) = 10.15216$$

$$\Omega_2^2 = (5.98756) \cos(18.16326 + 120) - (-4.46295) = 0.00193$$

$$\Omega_3^2 = (5.98756) \cos(18.16326 + 240) - (-4.46295) = 3.23476$$

The fundamental natural frequency is obtained from Eq. (15-7) as

$$\begin{aligned} \omega &= \{\Omega^2 E / [\rho(1 - \mu^2)r^2]\}^{1/2} \\ &= \{0.00193 \times 30,000,000 / [0.733 \times 10^{-3}(0.914)^4]\}^{1/2} \\ &= 4655.7 \text{ rad/sec} \\ &= 741 \text{ cps.} \end{aligned}$$

The natural frequency equations derived so far are for a cylinder with simply supported ends. The natural frequency values for a cylinder with fixed ends are normally larger in magnitude than those of a similar cylinder with simply supported edges as illustrated in Fig. 15-4.

Obtaining the natural frequency from Eq. (15-8) requires tedious calculations as demonstrated by the example above. Various simplifications of this equation

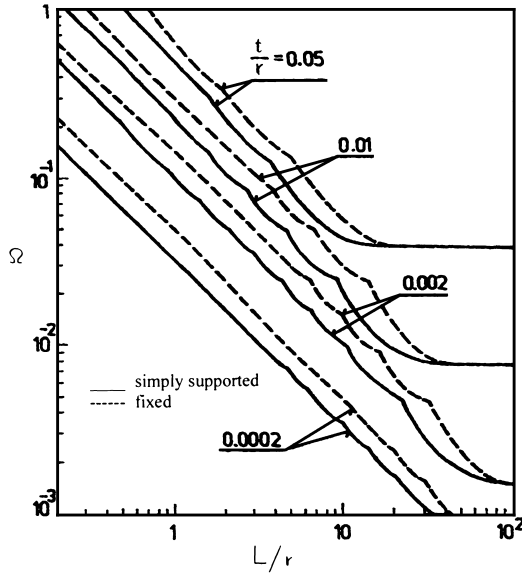


Figure 15-4. Fundamental Frequency parameter Ω as a Function of L/r for Simply Supported and Fixed Cylinders Using $\mu = 0.30$. (Markus, 1988).

have been proposed and they include neglecting the inertia forces in the x and θ directions as well as rotatory inertia (Seide 1975). For $n > 1$ a simplified expression for the natural frequency may be expressed as

$$\omega^2 = \frac{1}{r^2} \frac{E}{\rho(1-\mu^2)} \frac{1}{(\lambda^2 + n^2)^2} [\lambda^4(1-\mu^2) + k(\lambda^2 + n^2)^2(\lambda^2 + n^2 - 1)^2] \quad (15-9)$$

and for $\lambda^2 \ll 1$, Eq. (15-9) becomes

$$\omega^2 = \frac{1}{r^4} \frac{E}{\rho(1-\mu^2)} (0.083t^2)(n^2 - 1)^2 \quad (15-10)$$

Example 15-2

Use Eq. (15-9) to find the approximate fundamental natural frequency of the cylindrical shell in Example 15-1.

Solution:

$$\begin{aligned} \omega^2 &= (0.25)(44,975,488,359)(.01188)[(0.02802) \\ &\quad + (2.03151 \times 10^{-5})(84.18907)(66.83815) \\ &= (133,802,078)(0.02802 + 0.11448) \\ &= 19,067,162 \end{aligned}$$

$$\omega = 4366.6 \text{ radians/sec}$$

$$\omega = 695 \text{ cps}$$

This value is within 7% of the theoretical value in Example 15-1.

15-2 Spherical Shells

The differential equations for the free vibration of spherical shells are derived in a similar manner as those for cylindrical shells. Simplifications to the equations can be made for shallow shells where the angle ϕ , Fig. 15-5, is less than 30° . The resulting equation for the fundamental natural frequency can be expressed as

$$\omega = \frac{2.98(E/\rho)^{1/2}t}{R^2(1-\mu^2)^{1/2}} \{1 + [0.9 + 0.2(1+\mu)](1+\mu)(H/t)^2\}^{1/2} \quad (15-9)$$

The results obtained from this equation are very close (Kraus, 1967) to the results obtained from the equation derived from the non-shallow theory as long as the angle ϕ is equal to or less than 30° , Fig. 15-6.

Example 15-3

What is the natural frequency of the spherical roof shown in tank roof shown in Fig. 9-2a? Let $t = 0.25$ inch, $E = 30,000$ ksi, $\rho = 0.733 \times 10^{-3}$ lbs-sec²/in⁴, $R = 576$ inches, $H = 52.38$ inch, and $\mu = 0.3$.

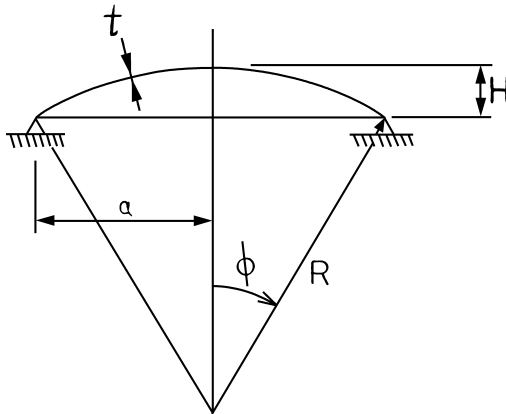


Figure 15-5. Shallow Spherical Shell.

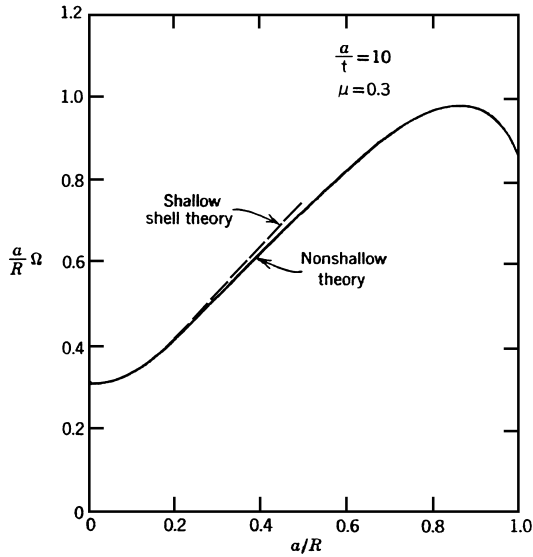


Figure 15-6. Fundamental Natural Frequency of Fixed Spherical Shells (Kraus, 1967).

Solution:

From Eq.(15-9),

$$\begin{aligned}\omega &= \frac{2.98(30,000,000/0.733 \times 10^{-3})^{1/2}(0.25)}{576^2(1 - 0.3^2)^{1/2}} \\ &\quad \times \left\{ 1 + [0.9 + 0.2(1 + 0.3)](1 + 0.3)(52.38/0.25)^2 \right\}^{1/2} \\ \omega &= \frac{150,717.9}{316,494.1} [1 + (1.16)(1.3)(43,898.6)]^{1/2} \\ \omega &= 122.5 \text{ radians/sec} \\ \omega &= 19.5 \text{ cps}\end{aligned}$$

The equations for the natural frequency of non-shallow spherical shells tend to be more complicated than that for shallow shells. A. Kalnins did extensive work in this area and he has published numerous articles (see for example Kalnins, 1964). The expression for the natural frequency of a complete sphere can be expressed as

$$\begin{aligned}2(1 - \mu^2)\Omega^2 &= 3(1 + \mu) + g_n + k(g_n + 3)(g_n + 1 + \mu) \\ &\quad \pm \left\{ [3(1 + \mu) + g_n + k(g_n + 3)(g_n + 1 + \mu)]^2 \right. \\ &\quad \left. - 4g_n[(1 - \mu^2) + k(g_n^2 + 2g_n + 1 - \mu^2)] \right\}^{1/2}\end{aligned}\quad (15-10)$$

Where,

$$\Omega^2 = \rho R^2 \omega^2 / E \quad (15-11)$$

and

$$g_n = n(n+1) - 2$$

$$k = \frac{t^2}{12R^2}$$

Example 15-4

What is the natural frequency of a spherical tank with $t = 0.25$ inch, $E = 30,000$ ksi, $\rho = 0.733 \times 10^{-3}$ lbs-sec²/in⁴, $R = 576$ inches, and $\mu = 0.3$.

Solution:

Three values of n are tried in order to obtain the minimum natural frequency. They are $n = 1, 2$, and 3 . It was found that the minimum natural frequency is obtained when $n = 2$. The following calculations are based on $n = 2$.

$$k = 1.56983 \times 10^{-8}$$

$$\text{For } n = 2, g_n = 4,$$

From Eq. (15-10),

$$\begin{aligned} 2(0.91)\Omega^2 &= 3(1.3) + 4 + (1.56983 \times 10^{-8})(7)(5.3) \\ &\pm \left\{ [3(1.3) + 4 + (1.56983 \times 10^{-8})(7)(5.3)]^2 \right. \\ &\quad \left. - 4(4)[(0.91) + (1.56983 \times 10^{-8})(4^2 + 2(4) + 1 - 0.3^2)] \right\}^{1/2} \\ 1.82\Omega^2 &= 7.90000 \pm 6.91737 \end{aligned}$$

minimum value of $\Omega^2 = 0.53991$.

From Eq. (15-11),

$$0.53991 = (0.733 \times 10^{-3})(576)^2 \omega^2 / 30,000,000$$

or,

$$\omega^2 = 66,603$$

and

$$\omega = 258 \text{ radians/second}$$

$$\omega = 41.1 \text{ cps}$$

Basic Finite Element Equations

16-1 Definitions

The finite element method is a powerful tool for calculating stress in complicated shell and plate structures that are difficult to analyze by classical plate and shell theories. The method consists of subdividing a given domain into small elements connected at the nodal points as shown in Fig. 16-1. The mathematical formulation consists of combining the governing equations of each of the elements to form a solution for the domain that satisfies the boundary conditions. The approximations associated with finite element solutions depend on many variables such as the type of element selected, number of elements used to model the domain, and the boundary conditions.

The complete derivation of the various equations for one-, two- and three-dimensional elements is beyond the scope of this book. However, a few equations are derived here to demonstrate the basic concept of Finite Element formulation and its applicability to the solution of plates and shells.

We begin the derivations by defining various elements (Fig. 16-2) and terms. Figure 16-2a shows a one-dimensional element in the x -direction with two nodal points, i and j . Figure 16-2b shows a two-dimensional triangular element in the x, y plane with nodal points i, j , and k . And Fig. 16-2c shows a three-dimensional rectangular brick element with eight nodal points.

Let the matrix $[\delta]$ define the displacements within an element. The size of the displacement matrix (Weaver and Johnston, 1984) depends on the complexity of the element being considered. The matrix $[q]$ defines nodal point displacements of an element and matrix $[F]$ defines the applied loads at the nodal points. The size of matrices $[q]$ and $[F]$ depends on the type and geometry of the element being considered. For the one-dimensional element in Fig. 16-2a, the quantities $[q]$ and $[F]$ are defined as

$$\begin{aligned} [q] &= \begin{bmatrix} q_{ix} \\ q_{jx} \end{bmatrix} \\ [F] &= \begin{bmatrix} F_{ix} \\ F_{jx} \end{bmatrix} \end{aligned} \quad (16-1)$$

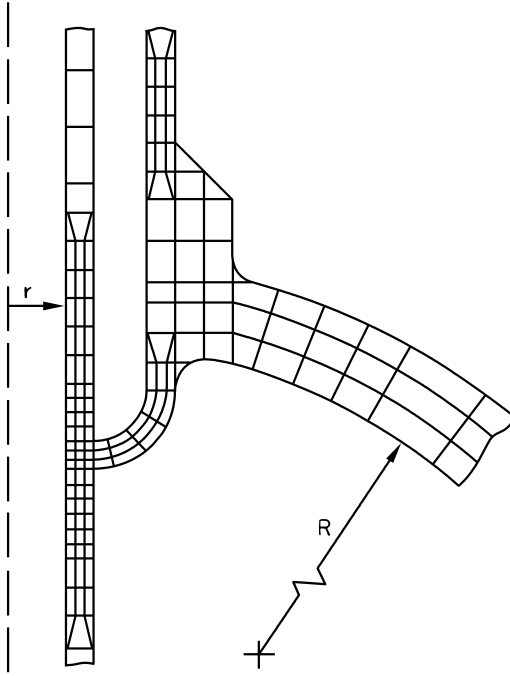


Figure 16-1. Nozzle-to-head attachment.

Similarly, the $[q]$ and $[F]$ matrices for the two-dimensional triangular element in Fig. 16-2b are expressed by

$$[q] = \begin{bmatrix} q_{ix} \\ q_{iy} \\ q_{jx} \\ q_{jy} \\ q_{kx} \\ q_{ky} \end{bmatrix} \quad [F] = \begin{bmatrix} F_{ix} \\ F_{iy} \\ F_{jx} \\ F_{jy} \\ F_{kx} \\ F_{ky} \end{bmatrix}. \quad (16-2)$$

The shape function matrix $[N]$ defines the relationship between a function at the nodal points and the same function within the element. Thus, the relationship between the nodal deflection $[q]$, Fig. 16-2a, and the general deflection δ at any point in the one-dimensional element is expressed as

$$\begin{aligned} \delta &= [N_i \quad N_j] \begin{bmatrix} q_i \\ q_j \end{bmatrix} \\ &= [N][q] \end{aligned}$$

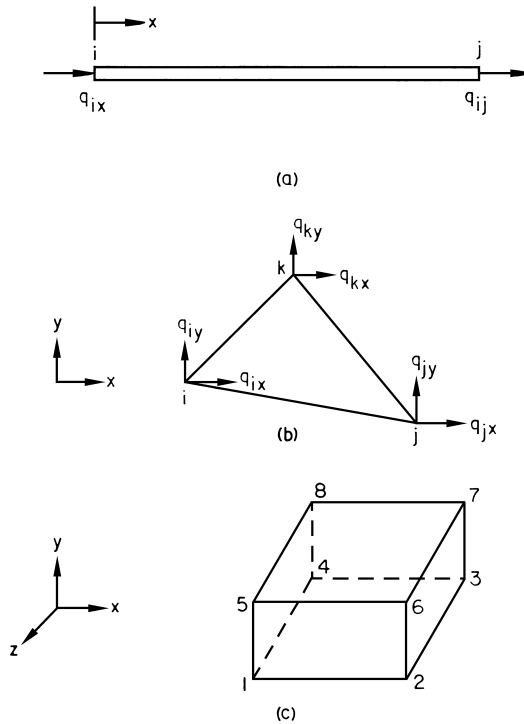


Figure 16-2. One, two, and three dimensional elements.

while the relationship between the nodal displacements $[q]$ and the general displacements $[\delta]$ in Fig. 16-2b for a two-dimensional element is

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_k & 0 \\ 0 & N_i & 0 & N_j & 0 & N_k \end{bmatrix} \begin{bmatrix} q_{ix} \\ q_{iy} \\ q_{jx} \\ q_{jy} \\ q_{kx} \\ q_{ky} \end{bmatrix}$$

or

$$[\delta] = [N][q]. \quad (16-3)$$

Let the strain-displacement matrix $[d]$ define the relationship between the strains in a continuum to the displacements in accordance with

$$[\varepsilon] = [d][\delta] \quad (16-4)$$

Strain in an element can also be expressed in terms of the deflection of the nodal points. Substituting Eq. (16-3) into Eq. (16-4) gives

$$[\varepsilon] = [d][N][q]. \tag{16-5}$$

Equation (14-5) can also be written as

$$[\varepsilon] = [B][q] \tag{16-6}$$

where

$$[B] = [d][N]. \tag{16-7}$$

The stress–strain relationship is obtained from Eq. (1-13) as

$$\left. \begin{aligned} [\sigma] &= [D][\varepsilon] - [D][\varepsilon_o] \\ &= [D][B][q] - [D][\varepsilon_o] \end{aligned} \right] \tag{16-8}$$

where $[\varepsilon_o]$ is the initial strain in a domain and $[\varepsilon]$ is the total strain.

With these definitions, the basic finite element equations can now be derived. Referring to Fig. 5-1, the strain energy for a differential element of volume dV is

$$dU = \frac{1}{2} [\varepsilon]^T [\sigma] - \frac{1}{2} [\varepsilon_o]^T [\sigma] \tag{16-9}$$

The total strain energy is given by

$$U = \int_V \frac{1}{2} ([\varepsilon]^T [\sigma] - [\varepsilon_o]^T [\sigma]) dV \tag{16-10}$$

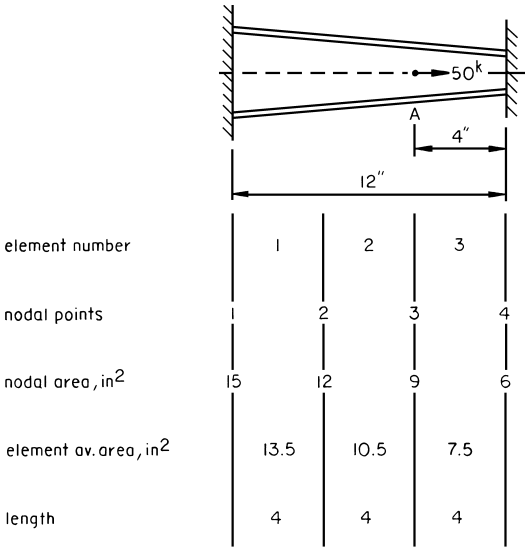


Figure 16-3. Conical shell.

Substituting Eq. (16-8) into Eq. (16-10) yields

$$U = \frac{1}{2} \left\{ [q]^T [B]^T [D] [B] [q] - 2 [q]^T [B]^T [D] [\varepsilon_0] + [\varepsilon_0]^T [D] [\varepsilon_0] \right\} dV. \quad (16-11)$$

The external work due to the nodal loads $[F]$ is

$$W_F = [F]^T [q]. \quad (16-12)$$

The external work due to surface pressure, $[p]$, is

$$W_p = ([u][p]) ds$$

or

$$W_p = \int_s ([q]^T [N]^T [p]) ds. \quad (16-13)$$

The potential energy of one element is

$$\Pi = U - (W_F + W_p)$$

or for the whole system

$$\Pi = \sum_{e=1}^E [U^e - (W_F^e + W_p^e)] \quad (16-14)$$

where e refers to any given element.

The minimum potential energy is obtained from

$$\frac{\partial \Pi}{\partial q} = 0$$

or,

$$\begin{aligned} \frac{\partial \Pi}{\partial q} = \sum_{e=1}^E & \left[\int_V [B_e]^T [D_e] [B_e] dV [q] \right. \\ & - \int_V [B_e]^T [D_e] [\varepsilon_0] dV \\ & \left. - \int_S [N_e]^T [p_e] ds \right] - F_e = 0. \end{aligned} \quad (16-15)$$

The quantity

$$[B_e]^T [D_e] [B_e] dV$$

is called the stiffness matrix of an element and is written as

$$[K_e] = \int_V [B_e]^T [D_e] [B_e] dV. \quad (16-16)$$

Hence, the finite element equation becomes

$$\begin{aligned} \sum_{e=1}^E [K_e] [q] = \sum_{e=1}^E & \left[\int_V [B_e]^T [D_e] [\varepsilon_0] dV \right. \\ & \left. + \int_S [N_e]^T [p_e] ds \right] + F_e \end{aligned} \quad (16-17)$$

which can be abbreviated as

$$[K_e][q] = [F] \quad (16-18)$$

where

$[F]$ = applied forces.

Equation (16-17) is the basic finite element equation for a domain.

Problems

- 16-1 Write the matrices $[q]$ and $[F]$ for the three-dimensional element in Fig. 16-2c.
 16-2 Derive Eq. (16-11).
 16-3 Derive Eq. (16-15).

16-2 One-Dimensional Elements

In formulating the finite element equations, the shape of the element as well as other functions such as applied loads, deflections, strains, and stresses are approximated by a polynomial. The size of the polynomial depends on the degrees of freedom at the nodal points and the accuracy required. Hence, for the one-dimensional element shown in Fig. 16-2a, a polynomial for a function such as deflection (Grandin 1986) may be expressed as

$$\delta = C_1 + C_2x \quad (16-19)$$

where

x = length along the x -axis;

C_1 and C_2 are constants.

The polynomial given by Eq. (16-19) can be written as a function of two matrices $[g]$ and $[C]$ as

$$\delta = [g][C] \quad (16-20)$$

$$\delta = [1 \quad x] \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

At the nodal points x_i and x_j , Eq. (16-20) becomes

$$\begin{bmatrix} q_i \\ q_j \end{bmatrix} = \begin{bmatrix} 1 & x_i \\ 1 & x_j \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \quad (16-21)$$

Define $[h]$ as the relationship between $[C]$ and $[q]$ at the nodal points. Thus, Eq. (16-21) becomes

$$\begin{bmatrix} q_i \\ q_j \end{bmatrix} = [h][C]$$

where

$$[h] = \begin{bmatrix} 1 & x_i \\ 1 & x_j \end{bmatrix}.$$

Solving for the matrix $[C]$ gives

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = [h]^{-1} \begin{bmatrix} q_i \\ q_j \end{bmatrix}$$

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{x_j - x_i} \begin{bmatrix} x_j & -x_i \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q_i \\ q_j \end{bmatrix} \quad (16-22)$$

or, from Eq. (16-20) the function at any point is

$$\delta = [g][h]^{-1} \begin{bmatrix} q_i \\ q_j \end{bmatrix}$$

or

$$\delta = \frac{1}{x_j - x_i} \begin{bmatrix} (x_j - x) & (-x_i + x) \end{bmatrix} \begin{bmatrix} q_i \\ q_j \end{bmatrix}. \quad (16-23)$$

The quantity $[g][h]^{-1}$ relates the deflection at the nodal points to that within the element. It is called the shape function and is designated as $[N]$. Thus,

$$[N] = [g][h]^{-1}. \quad (16-24)$$

Then Eq. (16-23) can be written as

$$\delta = [N] \begin{bmatrix} q_i \\ q_j \end{bmatrix} \quad (16-25)$$

where

$$[N] = \frac{1}{x_j - x_i} \begin{bmatrix} (x_j - x) & (-x_i + x) \end{bmatrix}. \quad (16-26)$$

Once the shape function for a linear element is established, the governing stiffness expression, Eq. (16-18), can also be determined. Thus, for the one-dimensional element shown in Fig. 16-2a, the general deflection function is expressed by Eq. (16-19) and the shape function $[N]$ is given by Eq. (16-26). From Hook's law, the strain in an axial member is expressed as

$$\varepsilon = \frac{d}{dx} u$$

and from Eqs. (16-5) and (16-6)

$$[\varepsilon] = \frac{d}{dx} [N][q]$$

or

$$[\varepsilon] = [B] \begin{bmatrix} q_i \\ q_j \end{bmatrix} \quad (16-27)$$

where

$$[B] = \frac{[-1 \quad 1]}{x_j - x_i} = \frac{1}{L} [-1 \quad 1].$$

For a uniaxial body without initial strain,

$$[\sigma] = [E][\varepsilon]$$

Hence,

$$[D] = [E]$$

From Eq. (16-16) the value of the stiffness matrix $[K]$ becomes

$$[K] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \quad (16-28)$$

From Eq. (16-17), the first term on the righthand side is due to the thermal effect and reduces to

$$\alpha EA(\Delta T) \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \quad (16-29)$$

The second term on the righthand side of Eq. (16-17) is for the surface loads. In this case, the surface loads can be applied only at the nodal points i and j . Hence, when p_x is applied at node i ,

$$\int_S [N]^T [p_x] ds = p_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} \int ds = p_x A_i \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (16-30)$$

When p_x is applied at node j ,

$$\int_S [N]^T [p_x] ds = p_x A_j \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (16-31)$$

The complete finite element equation for one-dimensional elements is obtained by combining Eqs. (16-28) through (16-31)

$$\frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} q_i \\ q_j \end{bmatrix} = \alpha EA(\Delta T) \begin{bmatrix} -1 \\ 1 \end{bmatrix} + A_i p_x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + A_j p_x \begin{bmatrix} 0 \\ 1 \end{bmatrix} + F$$

or

$$[K][q] = [F]. \quad (16-32)$$

Stress in the member is obtained from Eq. (16-8) as

$$\sigma = [E][B][q] = \frac{E}{L} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_i \\ q_j \end{bmatrix} - E\alpha(\Delta T). \quad (16-33)$$

Example 16-1

Find the stress in the tapered conical shell, Fig. 16-3. The shell is subjected to a force of 50 kips at point A . The cross sectional area of the cone increases from 6 square inches at the right end to 15 square inches at the left end. The shell is also subjected to a uniform decrease in temperature of 50°F . Assume the shell to be subdivided into three equal lengths and let the coefficient of thermal expansion be equal to 7×10^{-6} inch/inch/ $^\circ\text{F}$. Also, let the modulus of elasticity equal 30×10^6 psi and Poisson's ratio equal 0.3.

Solution

Element 1

The stiffness matrix from Eq. (16-28) is

$$[K_1] = \frac{\bar{A}E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^7 \begin{bmatrix} 10.125 & -10.125 \\ -10.125 & 10.125 \end{bmatrix}.$$

The thermal force from Eq. (16-29) is

$$\alpha \bar{A}E(-\Delta T) \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 10^3 \begin{bmatrix} 141.75 \\ -141.75 \end{bmatrix}.$$

and Eq. (16-18) gives

$$10^7 \begin{bmatrix} 10.125 & -10.125 \\ -10.125 & 10.125 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = 10^3 \begin{bmatrix} 141.75 \\ -141.75 \end{bmatrix}.$$

Element 2

The governing Eq. (16-18) for element 2 is

$$10^7 \begin{bmatrix} 7.875 & -7.875 \\ -7.875 & 7.875 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = 10^3 \begin{bmatrix} 110.25 \\ -110.25 \end{bmatrix}.$$

Element 3

$$[K_3] = \frac{\bar{A}E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^7 \begin{bmatrix} 5.625 & -5.625 \\ -5.625 & 5.625 \end{bmatrix}$$

The thermal force is

$$\alpha \bar{A}E(-\Delta T) \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 10^3 \begin{bmatrix} 78.75 \\ -78.75 \end{bmatrix}.$$

The nodal forces are

$$= 10^3 \begin{bmatrix} 50 \\ 0 \end{bmatrix}$$

and Eq. (16-18) becomes

$$10^7 \begin{bmatrix} 5.625 & -5.625 \\ -5.625 & 5.625 \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix} = 10^3 \begin{bmatrix} 128.75 \\ -78.75 \end{bmatrix}.$$

Table 16-1 Total stiffness and force matrices

<i>Stiffness Matrix [K]</i>				
F/q	1	2	3	4
1	10.125	−10.125		
2	−10.125	10.125		
		7.875	−7.875	
3		−7.875	7.875	
			5.625	−5.625
4			−5.625	5.625

<i>Load Matrix [F]</i>	
Node	Force
1	141.75
2	−141.75
	110.25
3	−110.25
	128.75
4	−78.75

Combining the matrices for elements 1, 2, and 3 (Table 16-1) gives

$$10^7 \begin{bmatrix} 10.125 & -10.125 & 0 & 0 \\ -10.125 & 18.000 & -7.875 & 0 \\ 0 & -7.875 & 13.500 & -5.625 \\ 0 & 0 & -5.625 & 5.625 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = 10^3 \begin{bmatrix} 141.75 \\ -31.50 \\ 18.50 \\ -78.75 \end{bmatrix}.$$

Because the deflections q_1 and q_4 are zero at the supports, we can delete the first and last rows and columns from the stiffness matrix and the above matrix reduces to

$$10^7 \begin{bmatrix} 18.000 & -7.875 \\ -7.875 & 13.500 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = 10^3 \begin{bmatrix} -31.50 \\ 18.50 \end{bmatrix}.$$

Solving for the values of q_2 and q_3 results in

$$\begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = \frac{1}{10^4} \begin{bmatrix} -1.545 \\ 0.469 \end{bmatrix}.$$

The strain expression in each element is given by Eq. (16-27) as

$$\begin{aligned}\varepsilon &= \frac{1}{L}(-q_i + q_j) \\ \varepsilon_1 &= -0.386E - 4 \\ \varepsilon_2 &= 0.504E - 4 \\ \varepsilon_3 &= -0.117E - 4.\end{aligned}$$

The stress is obtained from Eq. (16-33) as

$$\begin{aligned}\sigma &= E\varepsilon - \alpha E(-\Delta T) \\ &= 30 \times 10^6 \varepsilon + 10,500 \\ \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} &= \begin{bmatrix} 9340 \\ 12,010 \\ 10,150 \end{bmatrix}\end{aligned}$$

A classical theoretical solution of this simple problem can be obtained for comparison purposes. We can let the right end grow freely due to temperature and applied load. We then apply a load at the end to let the deflection at the right end be equal to zero. Using for the deflection due to loads the equation

$$\begin{aligned}w &= \int \frac{F dx}{EA_x} \\ &= \frac{F}{E} \int \frac{dx}{15 - \frac{9x}{L}}\end{aligned}$$

we get $q_2 = -1.574$ $q_3 = 0.449$

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} 9290 \\ 11,930 \\ 10,040 \end{bmatrix}$$

The calculated stress in Example 16-1 is different in each of the elements. This causes a discontinuity in stress at internal nodal points joining two elements. To overcome this, a procedure (Segerlind 1976), called the conjugate stress method is used to average the stresses at the nodal points. It calculates an approximate average stress value at the nodal points from the following equation

$$[Q][\bar{\sigma}] = [R] \quad (16-34)$$

where

$[Q]$ = a function of the $[N]$ matrix;

$[\bar{\sigma}]$ = stress at the nodal points, called conformal stress;

$[R]$ = a function of the element stress, called conjugate stress.

In accordance with the theory of conjugate stress approximations, the matrices $[Q]$ and $[R]$ for an element are calculated from the quantities

$$[Q] = \int_V [N]^T [N] dV \quad (16-35)$$

and

$$[R] = \int_V [\sigma][N]^T dV \quad (16-36)$$

where $[\sigma]$ is the stress in the element.

In many applications the member axis, which is used to determine the stiffness and load matrices, does not coincide with the global axis of the structure as illustrated in Fig. 16-4. In order to accommodate this condition, the member orientation with respect to the global axes needs to be taken into consideration. The resulting stiffness matrix (Wang 1986) is of the form

$$[K_G] = \frac{AE}{L} \begin{bmatrix} k_1 & k_2 & -k_1 & -k_2 \\ & k_3 & -k_2 & -k_3 \\ & & k_1 & k_2 \\ \text{symmetric} & & & k_3 \end{bmatrix} \quad (16-37)$$

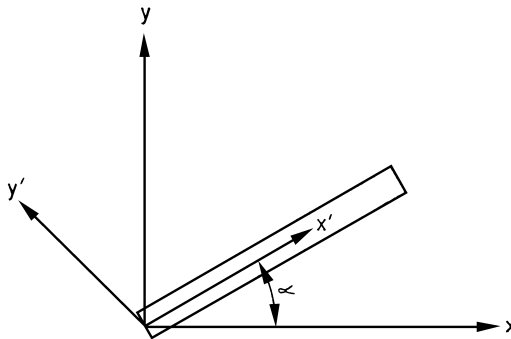


Figure 16-4. Element orientation.

where

$$k_1 = \cos^2 \alpha \quad k_2 = \cos \alpha \sin \alpha \quad k_3 = \sin^2 \alpha$$

α = angle shown in Fig. 16-4 and measured counterclockwise from the positive x -axis,

and load in each member is expressed as

$$F = \frac{EA}{L}(-q_{ix} \cos \alpha - q_{iy} \sin \alpha + q_{jx} \cos \alpha + q_{jy} \sin \alpha) \quad (16-38)$$

Problems

16-4 What is the thermal stress in the conical shell shown in Example 16-1 if the applied axial load is equal to zero?

16-5 What is the stress in the conical shell shown in Example 16-1 if the change in temperature is equal to zero?

16-3 Linear Triangular Elements

From Fig. 16-2b it is seen that each element has three nodal points and each nodal point has two degrees of freedom. Hence the displacement within the element is expressed by the following polynomial

$$u = C_1 + C_2x + C_3y$$

$$v = C_4 + C_5x + C_6y \quad (16-39)$$

where u and v are the deflection in the x - and y -axes, respectively. In matrix form these equations are written as

$$\begin{bmatrix} u \\ v \end{bmatrix} = [g][C]$$

where

$$[g] = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \quad (16-40)$$

and

$$[C] = \begin{bmatrix} C_1 \\ . \\ . \\ C_6 \end{bmatrix}.$$

The shape matrix $[N]$ is obtained from Eq. (16-24) and is expressed as

$$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_k & 0 \\ 0 & N_i & 0 & N_j & 0 & N_k \end{bmatrix} \quad (16-41)$$

where N_i , N_j , and N_k are defined as

$$N_i = \frac{1}{2\Delta} (a_i + b_i x + c_i y)$$

$$N_j = \frac{1}{2\Delta} (a_j + b_j x + c_j y)$$

$$N_k = \frac{1}{2\Delta} (a_k + b_k x + c_k y)$$

$$a_i = x_j y_k - x_k y_j, \quad b_i = y_j - y_k, \quad c_i = x_k - x_j$$

$$a_j = x_k y_i - x_i y_k, \quad b_j = y_k - y_i, \quad c_j = x_i - x_k$$

$$a_k = x_i y_j - x_j y_i, \quad b_k = y_i - y_j, \quad c_k = x_j - x_i$$

$$\Delta = \frac{1}{2} \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{vmatrix}$$

2Δ = area of triangle with coordinates $x_i y_i$, $x_j y_j$, $x_k y_k$.

The u and v expressions within the element are

$$\begin{bmatrix} u \\ v \end{bmatrix} = [N] \begin{bmatrix} q_{ix} \\ q_{iy} \\ q_{jx} \\ q_{jy} \\ q_{kx} \\ q_{ky} \end{bmatrix}.$$

The strain–deflection relationship is obtained from Eq. (2-16) as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \quad (16-42)$$

This strain expression can be designated as

$$[\varepsilon] = [d] \begin{bmatrix} u \\ v \end{bmatrix} = [d][N][q]$$

or

$$[\varepsilon] = [B][q]$$

where

$$[B] = [d][N] = \frac{1}{2\Delta} \begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ 0 & c_i & 0 & c_j & 0 & c_k \\ c_i & b_i & c_j & b_j & c_k & b_k \end{bmatrix}. \quad (16-43)$$

For a plane-stress formulation the stiffness matrix $[D]$ was derived in Chapter 1 and is given by Eq. (1-14) as

$$[D] = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}.$$

For a plane-strain formulation, Eq. (1-13) is used. Letting

$$\varepsilon_z = \gamma_{yz} = \gamma_{xz} = 0$$

$$[D] = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1 & \frac{\mu}{1-\mu} & 0 \\ \frac{\mu}{1-\mu} & 1 & 0 \\ 0 & 0 & \frac{1-2\mu}{2(1-\mu)} \end{bmatrix}. \quad (16-44)$$

The stiffness matrix is calculated from Eq. (16-16). The result can be expressed as

$$K = \frac{Et}{4\Delta} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ & & k_{33} & k_{34} & k_{35} & k_{36} \\ & & & k_{44} & k_{45} & k_{46} \\ & \text{symmetric} & & & k_{55} & k_{56} \\ & & & & & k_{66} \end{bmatrix} \quad (16-45)$$

where

$$\begin{aligned} k_{11} &= k_1(y_2 - y_3)^2 + k_3(x_3 - x_2)^2 \\ k_{12} &= k_2(x_3 - x_1)(y_2 - y_3) + k_3(x_3 - x_2)(y_2 - y_3) \\ k_{13} &= k_1(y_2 - y_3)(y_3 - y_1) + k_3(x_3 - x_2)(x_1 - x_3) \\ k_{14} &= k_2(x_1 - x_3)(y_2 - y_3) + k_3(x_3 - x_2)(y_3 - y_1) \\ k_{15} &= k_1(y_1 - y_2)(y_2 - y_3) + k_3(x_2 - x_1)(x_3 - x_2) \\ k_{16} &= k_2(x_2 - x_1)(y_2 - y_3) + k_3(x_3 - x_2)(y_1 - y_2) \\ k_{22} &= k_1(x_3 - x_2)^2 + k_3(y_2 - y_3)^2 \\ k_{23} &= k_2(x_3 - x_2)(y_3 - y_1) + k_3(x_1 - x_3)(y_2 - y_3) \\ k_{24} &= k_1(x_3 - x_2)(x_1 - x_3) + k_3(y_2 - y_3)(y_3 - y_1) \\ k_{25} &= k_2(x_3 - x_2)(y_1 - y_2) + k_3(x_2 - x_1)(y_2 - y_3) \\ k_{26} &= k_1(x_2 - x_1)(x_3 - x_2) + k_3(y_1 - y_2)(y_2 - y_3) \\ k_{33} &= k_1(y_3 - y_1)^2 + k_3(x_1 - x_3)^2 \\ k_{34} &= k_2(x_1 - x_3)(y_3 - y_1) + k_3(x_1 - x_3)(y_3 - y_1) \\ k_{35} &= k_1(y_1 - y_2)(y_3 - y_1) + k_3(x_1 - x_3)(x_2 - x_1) \\ k_{36} &= k_2(x_2 - x_1)(y_3 - y_1) + k_3(x_1 - x_3)(y_1 - y_2) \end{aligned}$$

$$k_{44} = k_1(x_1 - x_3)^2 + k_3(y_3 - y_1)^2$$

$$k_{45} = k_2(x_1 - x_3)(y_1 - y_2) + k_3(x_2 - x_1)(y_3 - y_1)$$

$$k_{46} = k_1(x_1 - x_3)(x_2 - x_1) + k_3(y_1 - y_2)(y_3 - y_1)$$

$$k_{55} = k_1(y_1 - y_2)^2 + k_3(x_2 - x_1)^2$$

$$k_{56} = k_2(x_2 - x_1)(y_1 - y_2) + k_3(x_2 - x_1)(y_1 - y_2)$$

$$k_{66} = k_1(x_2 - x_1)^2 + k_3(y_1 - y_2)^2$$

and

For Plane-Stress

$$k_1 = \frac{1}{1 - \mu^2} \quad k_2 = \frac{\mu}{1 - \mu^2} \quad k_3 = \frac{1}{2(1 + \mu)}$$

For Plane-Strain

$$k_1 = \frac{1 - \mu}{(1 + \mu)(1 - 2\mu)} \quad k_2 = \frac{\mu}{(1 + \mu)(1 - 2\mu)}$$

$$k_3 = \frac{1}{2(1 + \mu)}$$

The forces are calculated from Eq. (16-17).

Example 16-2

The triangular plate, Fig. 16-5a, is stiffened at the edges as shown. Find the stress in the various components. Let $E = 30,000$ ksi and $\mu = 0.3$.

Solution

The various nodal points are numbered as shown in Fig. 16-5b. The stiffness matrices, K , for members A , B , and C are obtained from Eq. (16-37) as

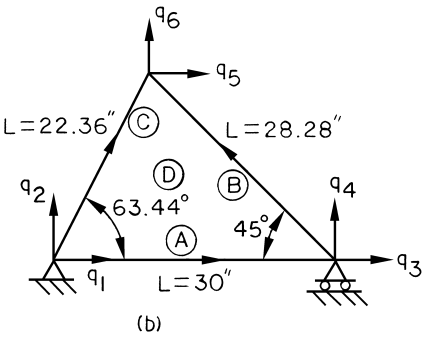
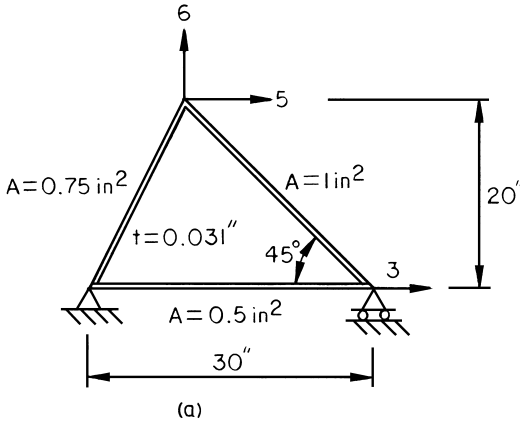


Figure 16-5. Stiffened triangular plate.

Member A with $\alpha = 0^\circ$

With nodal points q_1, q_2, q_3, q_4

$$K_A = \frac{0.5 \times 3000 \times 10^4}{30} \begin{bmatrix} 1.00 & 0.00 & -1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 \\ -1.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.00 \end{bmatrix}$$

or, since q_1, q_2 , and q_4 are zero, we eliminate rows and columns 1, 2, and 4 and we get

$$K_A = 10^4 [50].$$

Member B with $\alpha = 135^\circ$

With nodal points q_3, q_4, q_5, q_6

$$K_B = \frac{1 \times 3000 \times 10^4}{28.28} \begin{bmatrix} 0.50 & -0.50 & -0.50 & 0.50 \\ -0.50 & 0.50 & 0.50 & -0.50 \\ -0.50 & 0.50 & 0.50 & -0.50 \\ 0.50 & -0.50 & -0.50 & 0.50 \end{bmatrix}$$

or, since q_4 is zero,

$$K_B = 10^4 \begin{bmatrix} 53.04 & -53.04 & 53.04 \\ -53.04 & 53.04 & -53.04 \\ 53.04 & -53.04 & 53.04 \end{bmatrix}$$

Member C with $\alpha = 63.44^\circ$

With nodal points q_1, q_2, q_5, q_6

$$K_C = \frac{0.75 \times 3000 \times 10^4}{22.36} \begin{bmatrix} 0.20 & 0.40 & -0.20 & -0.40 \\ 0.40 & 0.80 & -0.40 & -0.80 \\ -0.20 & -0.40 & 0.20 & 0.40 \\ -0.40 & -0.80 & 0.40 & 0.80 \end{bmatrix}$$

or, since q_1 and q_2 are zero

$$K_C = 10^4 \begin{bmatrix} 20.125 & 40.25 \\ 40.125 & 80.50 \end{bmatrix}$$

Member D

With q_1, q_2 , and q_4 equal to zero

From Eq. (16-45)

$$K_D = \frac{Et}{4\Delta} \begin{bmatrix} K_{33} & K_{35} & K_{36} \\ & K_{55} & K_{56} \\ & & K_{66} \end{bmatrix}$$

$$K_D = 10^4 \begin{bmatrix} 74.11 & -17.90 & 30.69 \\ -17.90 & 53.71 & 0.00 \\ 30.69 & 0.00 & 153.31 \end{bmatrix}$$

From Table 16-2, the total matrix is

$$K = 10^4 \begin{bmatrix} 177.15 & -70.94 & 83.73 \\ -70.94 & 126.88 & -12.79 \\ 83.73 & -12.79 & 286.85 \end{bmatrix}$$

and the force matrix is

$$F = \begin{bmatrix} 3.0 \\ 5.0 \\ 7.0 \end{bmatrix} \text{ kips.}$$

From Eq. (16-32),

$$Kq = F$$

or

$$\begin{bmatrix} q_3 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 3.235 \\ 5.927 \\ 1.760 \end{bmatrix} \times 10^{-6} \text{ inch.}$$

The stresses in members *A*, *B*, and *C* are obtained from Eq. (16-38) as

$$\sigma_A = 3.24 \text{ ksi}, \quad \sigma_B = -0.7 \text{ ksi}, \quad \sigma_C = 5.67 \text{ ksi.}$$

Table 16-2. Total stiffness matrix (multiplied by 10⁴)

F/q	3	5	6
3	50.00		
	53.04	-53.04	53.04
	74.11	-17.90	30.69
5	-53.04	53.04	-53.04
		20.13	40.25
	-17.90	53.71	0.00
6	53.04	-53.04	53.04
		40.25	80.50
	30.69	0.0	153.31

The stress in plate D is obtained from Eq. (16-8) as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} 8.85 \\ 7.94 \\ 5.59 \end{bmatrix} \text{ ksi.}$$

16-4 Axisymmetric Triangular Linear Elements

Many plate and shell configurations, Fig. 16-1, are modeled as axisymmetric triangular elements, Fig. 16-6, have the same size N matrix as that defined by Eq. (16-41) for plane elements. The strain-stress relationship given by Eq. (16-42) for plane elements must be modified for axisymmetric elements to include the hoop strain ϵ_θ . Thus, Eq. (16-42) becomes

$$\begin{bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial r} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad (16-46).$$

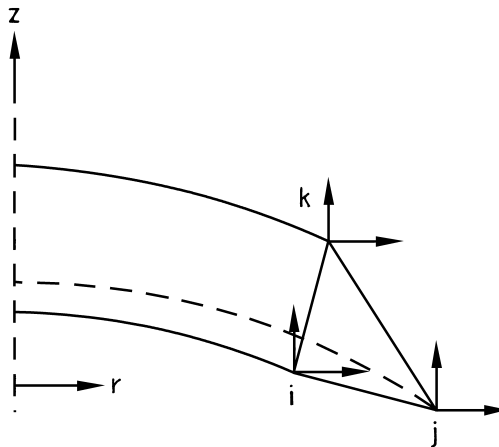


Figure 16-6. Triangular element.

and the $[B]$ matrix becomes

$$[B] = \frac{1}{2\Delta} \begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ 0 & c_i & 0 & c_j & 0 & c_k \\ \frac{2\Delta N_i}{r} & 0 & \frac{2\Delta N_j}{r} & 0 & \frac{2\Delta N_k}{r} & 0 \\ c_i & b_i & c_j & b_j & c_k & b_k \end{bmatrix} \quad (16-47)$$

where

$$\begin{bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{rz} \end{bmatrix} = [B] \begin{bmatrix} q_{ir} \\ q_{iz} \\ q_{jr} \\ q_{jz} \\ q_{kr} \\ q_{kz} \end{bmatrix} \quad (16-48)$$

The stress-strain relationship is obtained from

$$[D] = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1 & \frac{\mu}{1-\mu} & \frac{\mu}{1-\mu} & 0 \\ \frac{\mu}{1-\mu} & 1 & \frac{\mu}{1-\mu} & 0 \\ \frac{\mu}{1-\mu} & \frac{\mu}{1-\mu} & 1 & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2(1-\mu)} \end{bmatrix} \quad (16-49)$$

The stiffness matrix is determined from Eq. (16-16) as

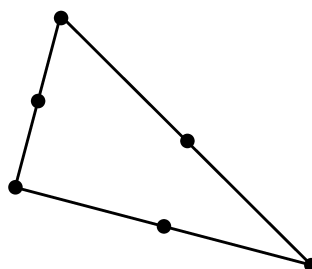
$$[K_e] = \int_v [B_e]^T [D_e] [B_e] dV.$$

The evaluation of the integral $(B^T D B) dV$ in axisymmetric problems is complicated by the fact that the matrix $[B]$ contains the variable $1/r$. A common procedure for integrating this quantity (Zienkiewicz 1977) is to use the radius $\bar{r}$ at the centroid of the element. Also, we can substitute for the quantity dV the value $(2\pi\bar{r}A)$ where A is the area of the element. Hence, the stiffness matrix $[K]$ becomes

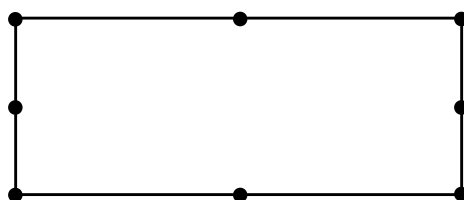
$$[K] = [B]^T [D] [B] 2\bar{r}A. \quad (16-50)$$



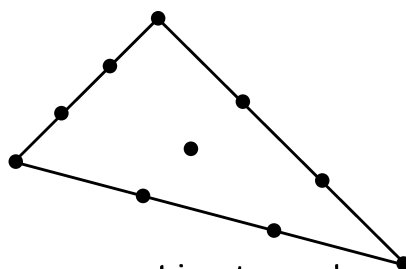
linear element



quadratic element



quadratic element



cubic element

Figure 16-7. Various elements.

16-5 Higher Order Elements

Equations derived for the linear triangular elements in Sections 16-3 and 16-4 can also be established for the linear rectangular elements shown in Fig. 16-7. The equations for the rectangular elements are slightly more complicated than those for triangular elements (Rockey et al. 1975) due to the additional fourth nodal point. In both cases the strain is constant throughout the element. This is a disadvantage in areas where a large strain gradient exists because a large number of elements is needed. Accordingly, higher order elements are normally utilized. The higher order elements have additional nodal points in the sides, and sometimes in the interior. With more nodal points in an element, the strain becomes more complex within an element and fewer elements are needed to define a complex geometry or an area with large strain gradients.

The shape function $[N]$ needed to define higher order elements is more complicated than that of linear elements and its derivation requires more sophisticated methods using natural coordinate systems (Weaver and Johnston 1984). Also, the stiffness matrix, which is a function of $[N]$, requires numerical integration which is cumbersome to evaluate without a computer. The accuracy of the results depends, in part, on the method used for the numerical integration.

Finite element formulation of a plate element, as well as finite element formulation of a shell element, have also been derived (Gallagher 1975) and are based on various polynomial approximations. The accuracy of these formulations depends on the particular plate or shell theory being used.

Finite element formulation of three-dimensional brick elements is also available in the literature. The equations become cumbersome for elements higher than quadratic.

APPENDIX A

Fourier Series

A-1 General Equations

A periodic function (Wylie 1960) can be represented by a series that is expressed as

$$f(x) = 0.5A_0 + A_1 \cos x + A_2 \cos 2x + \cdots + A_m \cos mx + B_1 \sin x \\ + B_2 \sin 2x + \cdots + B_m \sin mx$$

or

$$f(x) = 0.5A_0 + \sum_{m=1}^{\infty} A_m \cos mx + \sum_{m=1}^{\infty} B_m \sin mx. \quad (\text{A-1})$$

The series given by Eq. (A-1) is known as a Fourier Series and is used to express periodic functions such as those shown in Fig. A-1. The coefficients A and B in Eq. (A-1) are evaluated over a 2π period starting at a given point d . The value of A_0 can be obtained by integrating Eq. (A-1) from $x = d$ to $x = d + 2\pi$.

Thus,

$$\int_d^{d+2\pi} f(x) dx = 0.5A_0 \int_d^{d+2\pi} dx + A_1 \int_d^{d+2\pi} \cos x dx + \cdots \\ + A_m \int_d^{d+2\pi} \cos mx dx + B_1 \int_d^{d+2\pi} \sin x dx + \cdots \\ + B_m \int_d^{d+2\pi} \sin mx dx.$$

The first term in the righthand side of the equation gives πA_0 . All other terms on the righthand side are zero because of the relationships

$$\int_d^{d+2\pi} \cos mx dx = 0 \quad m \neq 0 \\ \int_d^{d+2\pi} \sin mx dx = 0.$$

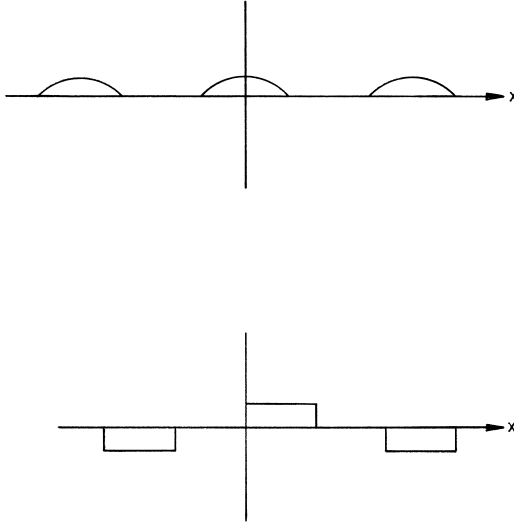


Figure A-1. Periodic functions.

Hence,

$$A_0 = \frac{1}{\pi} \int_d^{d+2\pi} f(x) dx. \quad (\text{A-2})$$

The A_m term in Eq. (A-1) can be obtained by multiplying both sides of the equation by $\cos mx$.

$$\begin{aligned} \int_d^{d+2\pi} f(x) \cos mx dx &= \frac{1}{2} A_0 \int_d^{d+2\pi} \cos mx dx + A_1 \int_d^{d+2\pi} \cos x \cos mx dx + \cdots \\ &+ A_m \int_d^{d+2\pi} \cos mx \cos mx dx + B_1 \int_d^{d+2\pi} \sin x \cos mx dx \\ &+ \cdots + B_m \int_d^{d+2\pi} \sin mx \cos mx dx. \end{aligned} \quad (\text{A-3})$$

Since

$$\int_d^{d+2\pi} \cos mx \cos nx dx = 0 \quad m \neq n$$

$$\int_d^{d+2\pi} \cos^2 mx dx = \pi \quad m \neq 0$$

$$\int_d^{d+2\pi} \cos mx \sin nx dx = 0.$$

Equation (A-3) becomes

$$\int_d^{d+2\pi} f(x) \cos mx \, dx = A_m \pi$$

or

$$A_m = \frac{1}{\pi} \int_d^{d+2\pi} f(x) \cos mx \, dx. \quad (\text{A-4})$$

Similarly the values of B_m can be found by multiplying both sides of Eq. (A-1) by $\sin mx$. Using the expressions

$$\int_d^{d+2\pi} \sin mx \sin nx \, dx = 0 \quad m \neq n$$

and

$$\int_d^{d+2\pi} \sin^2 mx \, dx = \pi$$

the equation becomes

$$B_m = \frac{1}{\pi} \int_d^{d+2\pi} f(x) \sin mx \, dx. \quad (\text{A-5})$$

Accordingly, we can state that for a given periodic function $f(x)$, a Fourier expansion can be written as shown in Eq. (A-1) with the various constants obtained from Eqs. (A-2), (A-4), and (A-5).

Example A-1

Express the function shown in Fig. A-2 in a Fourier Series

Solution

$$f(x) = 0 \quad -\pi < x < 0$$

$$f(x) = p_0 \quad 0 < x < \pi$$

$$d = -\pi$$

From Eq. (A-2),

$$A_0 = \frac{1}{\pi} \int_{-\pi}^0 (0) \, dx + \frac{1}{\pi} \int_0^{\pi} p_0 \, dx$$

or

$$A_0 = p_0.$$

From Eq. (A-4)

$$A_m = \frac{1}{\pi} \int_0^{\pi} p_0 \cos mx \, dx$$

$$A_m = 0.$$

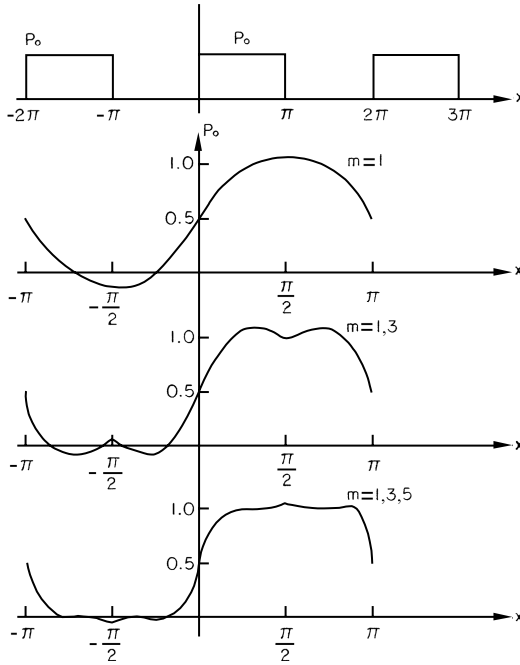


Figure A-2. Periodic rectangular function.

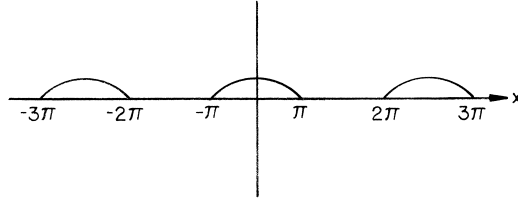
From Eq. (A-5),

$$\begin{aligned}
 B_m &= \frac{1}{\pi} \int_0^{\pi} p_0 \sin mx \, dx \\
 &= \frac{p_0}{m\pi} (-\cos mx) \Big|_0^{\pi} \\
 &= \frac{-p_0}{m\pi} (\cos m\pi - 1) \\
 B_m &= \frac{2p_0}{m\pi} \quad \text{when } m \text{ is odd} \\
 &= 0 \quad \text{when } m \text{ is even.}
 \end{aligned}$$

Therefore, the expansion of the function shown in Fig. A-2 is expressed as

$$f(x) = 0.5p_0 + \frac{2p_0}{\pi} \sum_{m=1,3,\dots}^{\infty} \frac{1}{m} \sin mx.$$

A plot of this equation with $m = 1, 3, 5$ is shown in Fig. A-2.



Prob. A-1. Periodic sinusoidal function.

Problems

A-1 What is the Fourier expansion of the function shown?

A-2 What is the Fourier expansion of the function shown?

A-2 Interval Change

In applying the Fourier Series to plate and shell problems, it is more convenient to specify intervals other than 2π . Defining the new interval as $2p$, Eqs. (A-2), (A-4), and (A-5) can be written as

$$A_0 = \frac{1}{p} \int_d^{d+2p} f(x) dx \quad (\text{A-6})$$

$$A_m = \frac{1}{p} \int_d^{d+2p} f(x) \cos \frac{m\pi x}{p} dx \quad (\text{A-7})$$

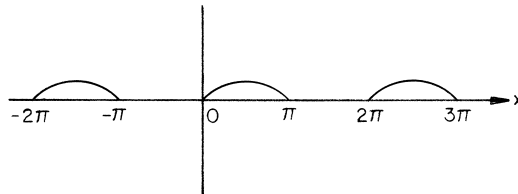
$$B_m = \frac{1}{p} \int_d^{d+2p} f(x) \sin \frac{m\pi x}{p} dx \quad (\text{A-8})$$

where

$2p$ = period of function,

and the series can be written as

$$f(x) = \frac{1}{2} A_0 + \sum_{m=1}^{\infty} A_m \cos \frac{m\pi x}{p} + \sum_{m=1}^{\infty} B_m \sin \frac{m\pi x}{p}. \quad (\text{A-9})$$



Prob. A-2. Periodic non-symmetric sinusoidal function.

Table A-1. Program “SNGLSUM”

```

10 REM THE NAME OF THIS PROGRAM IS SINGLE SUMMATION, "SNGLSUM"
20 REM THIS PROGRAM CALCULATES THE SUM OF A SINGLE SERIES
30 REM INPUT EQUATION TO BE EVALUATED IN LINE 210
40 REM STATEMENTS 210 TO 219 ARE RESRV'D FOR EQUAT'N TO BE SOLV'D
50 REM INPUT THE MAXIMUM SUMMATION VALUE IN LINE 200
60 PRINT "IS THE M SUMMATION ODD (O), EVEN (E), OR CONTINUOUS (C)"
70 INPUT A$
80 IF A$ = "O" THEN 120
90 IF A$ = "E" THEN 150
100 IF A$ = "C" THEN 180
110 GOTO 60
120 M1 = 1
130 M2 = 2
140 GOTO 200
150 M1 = 2
160 M2 = 2
170 GOTO 200
180 M1 = 1
190 M2 = 1
200 FOR M = M1 TO 30 STEP M2
210 Y = 1/(M)^2
220 LET T = T + Y
230 NEXT M
240 PRINT "SUMMATION OF SERIES IS = "; T
250 END

```

Parts of the equation under the summation signs can be solved by available computer programs. A short computer program called “SNGLSUM” is listed in Table A-1.

Example A-2

Find the Fourier expansion of the Function $f(x) = \cos x$ as shown in Fig. A-3.

Solution

The period $2p$ is equal to π . Thus, $p = \pi/2$ and $d = -\pi/2$.

$$A_0 = \frac{1}{\pi/2} \int_{-\pi/2}^{\pi/2} \cos x \, dx = 4/\pi$$

$$A_m = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \cos x \cos \frac{m\pi x}{\pi/2} \, dx = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(4m^2 - 1)}$$

$$B_m = 0.$$

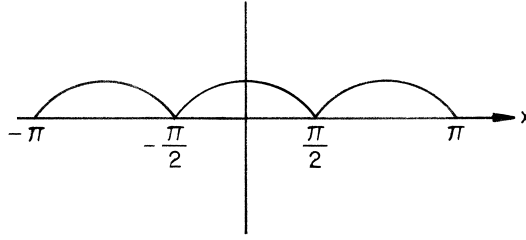


Figure A-3. Periodic cosine function.

and from Eq. (A-9),

$$f(x) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{4m^2 - 1} \cos 2mx.$$

A-3 Half-Range Expansions

If a function is symmetric with respect to the axis of reference as shown in Fig. A-4, then the coefficient integral can be simplified by integrating over one-half the period. This integration can be performed as an even or an odd function. Hence,

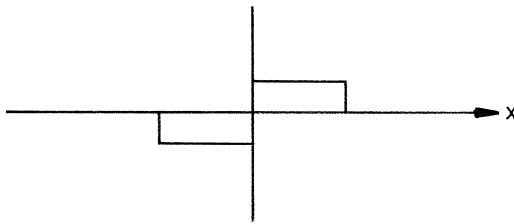
For an even periodic function

$$\left. \begin{aligned} A_0 &= \frac{2}{p} \int_0^p f(x) dx \\ A_m &= \frac{2}{p} \int_0^p f(x) \cos \frac{m\pi x}{p} dx \\ B_m &= 0. \end{aligned} \right\} \quad (\text{A-10})$$

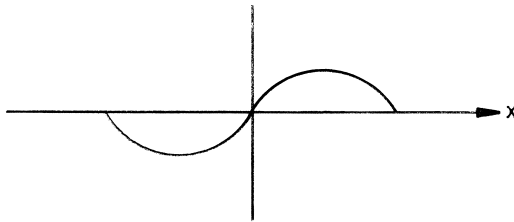
For an odd periodic function

$$\left. \begin{aligned} A_0 &= A_m = 0 \\ B_m &= \frac{2}{p} \int_0^p f(x) \sin \frac{m\pi x}{p} dx \end{aligned} \right\} \quad (\text{A-11})$$

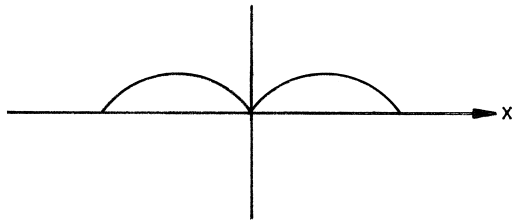
It should be noted that the even and odd functions defined by Eqs. (A-10) and (A-11) and Fig. A-4 do not refer necessarily to the shape of the function but rather to the reference line from which they are defined. This can best be illustrated by the following example.



(a) odd



(b) odd



(c) even

Figure A-4. Periodic symmetric function.

Example A-3

Figure A-5 shows a plot of the function $y = x - x^2$. Obtain and plot the Fourier Series expansion of this function (a) from $y = -1$ to $y = 1$; (b) as an even series from $y = 0$ to $y = 1$; (c) as an odd series from $y = 0$ to $y = 1$.

Solution

(a)

$$d = -1, \quad 2p = 2 \quad \text{or} \quad p = 1$$

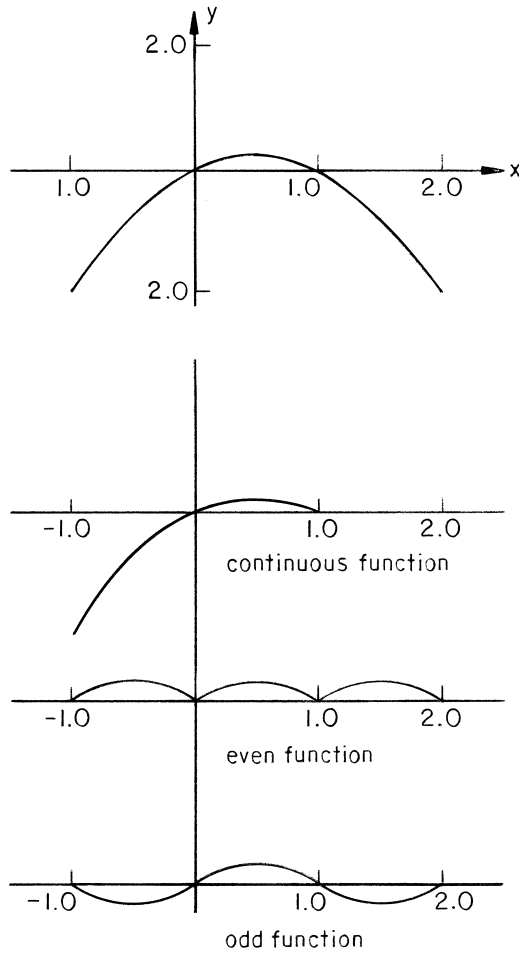


Figure A-5. Function $y = x - x^2$.

$$A_0 = \int_{-1}^1 (x - x^2) dx = -2/3$$

$$A_m = \int_{-1}^1 (x - x^2) \cos \frac{m\pi x}{1} dx = -\frac{4 \cos m\pi}{m^2 \pi^2}$$

$$B_m = \int_{-1}^1 (x - x^2) \sin \frac{m\pi x}{1} dx = -\frac{2 \cos m\pi}{m\pi}$$

$$\begin{aligned}
 f(x) &= -1/3 - \sum_{m=1}^{\infty} \frac{4 \cos m\pi}{m^2 \pi^2} \cos m\pi x - \sum_{m=1}^{\infty} \frac{2 \cos m\pi}{m\pi} \sin m\pi x \\
 &= -1/3 - \frac{4}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2} \cos m\pi x - \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin m\pi x
 \end{aligned}$$

(b)

$$A_0 = 2 \int_0^1 (x - x^2) dx = 1/3$$

$$A_m = 2 \int_0^1 (x - x^2) \cos \frac{m\pi x}{1} dx = -\frac{2(1 + \cos m\pi)}{m^2 \pi^2}$$

$$B_m = 0$$

$$f(x) = 1/6 - \sum_{m=1}^{\infty} \frac{2(1 + \cos m\pi)}{m^2 \pi^2} \cos m\pi x$$

(c)

$$A_0 = A_m = 0$$

$$B_m = 2 \int_0^1 (x - x^2) \sin \frac{m\pi x}{1} dx = \frac{4(1 - \cos m\pi)}{m^3 \pi^3}$$

$$f(x) = \sum_{m=1}^{\infty} \frac{4(1 - \cos m\pi)}{m^3 \pi^3} \sin m\pi x$$

A-4 Double Fourier Series

In solving rectangular plate problems of length a and width b , it is customary to express the applied loads in terms of a single or double series. The double Fourier series is normally expressed as an odd periodic function with a half range period given between 0 and a for one side of the plate and 0 to b for the other side. Thus,

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (\text{A-12})$$

where

$$B_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy. \quad (\text{A-13})$$

Equation (A-12) can be solved by many available commercial programs. One simplified such program, called “DBLSUM,” is listed in Table A-2.

Table A-2. Program “DBLSUM”

```

10 REM THE NAME OF THIS PROGRAM IS DOUBLE SUMMATION, “DBLSUM”
20 REM THIS PROGRAM CALCULATES THE SUM OF A DOUBLE SERIES
30 REM INPUT EQUATION TO BE EVALUATED IN LINE 370
40 REM STATEMENTS 370 TO 379 ARE RESRV'D FOR EQUAT'N TO BE SOLV'D
45 REM LINES 350 AND 360 GIVE THE MAX. SUMMATION VALUE
50 PRINT “IS THE M SUMMATION ODD (O), EVEN (E), OR CONTINUOUS (C)”
60 INPUT A$
70 IF A$ = “O” THEN 110
80 IF A$ = “E” THEN 140
90 IF A$ = “C” THEN 170
100 GOTO 50
110 M1 = 1
120 M2 = 2
130 GOTO 200
140 M1 = 2
150 M2 = 2
160 GOTO 200
170 M1 = 1
180 M2 = 1
190 GOTO 200
200 PRINT “IS THE N SUMMATION ODD (O), EVEN (E), OR CONTINUOUS (C)”
210 INPUT A$
220 IF A$ = “O” THEN 260
230 IF A$ = “E” THEN 290
240 IF A$ = “C” THEN 320
250 GOTO 200
260 N1 = 1
270 N2 = 2
280 GOTO 350
290 N1 = 2
300 N2 = 2
310 GOTO 350
320 N1 = 1
330 N2 = 1
340 GOTO 350
350 FOR M = M1 TO 30 STEP M2
360 FOR N = N1 TO 30 STEP N2
370  $U = (-1)^{(M+N)/2 - 1}$ 
371  $V = M*N*(M^2 + N^2)^2$ 
372 Y = U/V
380 LET T = T + Y
390 NEXT N
400 NEXT M
410 PRINT “SUMMATION OF SERIES IS = ”; T
420 END

```

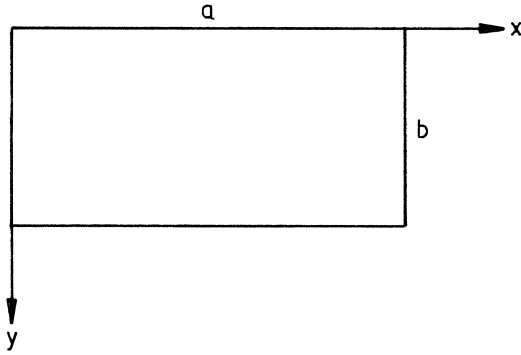


Figure A-6. Rectangular plate.

Example A-4

The rectangular plate shown in Fig. A-6 is subjected to a uniform pressure p_o . Determine the Fourier expansion for the pressure.

Solution

From Eq. (A-13),

$$\begin{aligned}
 B_{mn} &= \frac{4p_o}{ab} \int_0^b \int_0^a \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy \\
 &= \frac{16p_o}{\pi^2 mn} \quad m, n \text{ are odd functions} \\
 f(x, y) &= \frac{16p_o}{\pi^2} \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} \frac{1}{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
 \end{aligned}$$

APPENDIX B

Bessel Functions

B-1 General Equations

In many plate and shell applications involving circular symmetry, the resulting differential equations are solved by means of a power series known as Bessel functions. Some of these functions are discussed in this appendix.

The differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + y = 0 \quad (\text{B-1})$$

is referred to as Bessel's equation of zero order. Its solution (Bowman 1958) is given by the following power series

$$y = C_1 J_0(x) + C_2 Y_0(x) \quad (\text{B-2})$$

where

C_1 and C_2 = constants obtained from boundary conditions; $J_0(x)$ = Bessel function of the first kind of zero order.

$$\begin{aligned} J_0(x) &= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m} \end{aligned}$$

$Y_0(x)$ = Bessel function of the second kind of zero order.

$$\begin{aligned} Y_0(x) &= J_0(x) \int \frac{dx}{x J_0^2(x)} \\ &= J_0(x) \ln x + \frac{x^2}{2^2} - \frac{x^4}{2^2 \cdot 4^2} (1 + 1/2) + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} (1 + 1/2 + 1/3) - \dots \end{aligned}$$

Equation (B-1) is usually encountered in a more general form as

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - k^2)y = 0. \quad (\text{B-3})$$

The solution of this equation (Hildebrand 1964) is

$$y = C_1 J_k(x) + C_2 J_{-k}(x)$$

when k is not zero or a positive integer, or

$$y = C_1 J_k(x) + C_2 Y_k(x)$$

when k is zero or a positive integer. And where,

$J_k(x)$ = Bessel function of the first kind of order k

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)(m+k)!} (x/2)^{2m+k}$$

$J_{-k}(x)$ = Bessel function of the first kind of order k

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)(m-k)!} (x/2)^{2m-k}$$

$Y_k(x)$ = Bessel function of the second kind of order k

$$= \frac{2}{\pi} \left[(\ln(x/2) + \gamma) J_k(x) - \frac{1}{2} \sum_{m=0}^{k-1} \frac{(k-m-1)!}{m!} (x/2)^{2m-k} \right. \\ \left. + \frac{1}{2} \sum_{m=0}^{\infty} (-1)^{m+1} [h(m) + h(m+k)] \frac{(x/2)^{2m+k}}{m!(m+k)!} \right]$$

$$h(m) = \sum_{r=1}^m 1/r \quad m > 1.$$

$$\gamma = 0.5772$$

A plot of $J_0(x)$, $J_1(x)$, $Y_0(x)$, and $Y_1(x)$ is shown in Fig. B-1. Also, Table B-1 gives some values of $J(x)$ and $Y(x)$.

A different form of Eq. (B-3) that is encountered often in plate and shell theory is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (x^2 + k^2)y = 0. \quad (\text{B-4})$$

The solution of this equation (Dwight 1972) is

$$y = C_1 I_k(x) + C_2 I_{-k}(x)$$

when k is not zero or a positive integer, or

$$y = C_1 I_k(x) + C_2 K_k(x)$$

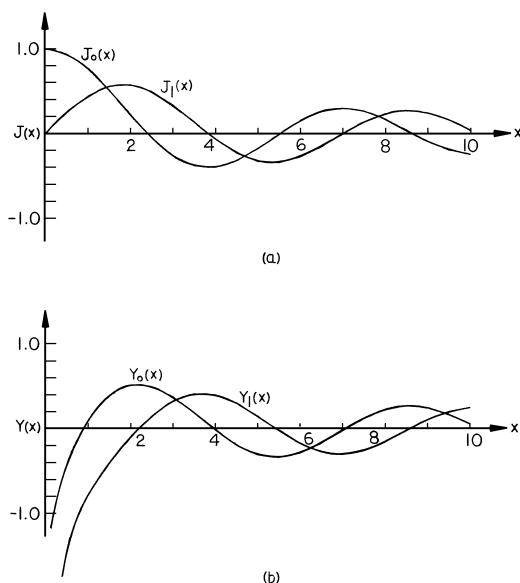


Figure B-1. A plot of various Bessel functions.

when k is zero or a positive integer and where

$I_k(x)$ = modified Bessel function of the first kind of order k

$$= \sum_{m=1}^{\infty} \frac{(x/2)^{2m+k}}{m!(m+k)!}.$$

$K_k(x)$ = modified Bessel function of the second kind of order k

$$\begin{aligned} &= (-1)^{k+1} [\ln(x/2) + \gamma] I_k(x) \\ &+ \frac{1}{2} \sum_{m=0}^{k-1} \frac{(-1)^m (k-m-1)!}{m!} (x/2)^{2m-k} \\ &+ \frac{1}{2} \sum_{m=0}^{\infty} \frac{(-1)^k (x/2)^{2m+k}}{m!(m+k)!} [(1 + 1/2 + \cdots + 1/m) \\ &+ (1 + 1/2 + \cdots + 1/(m+k))]. \end{aligned}$$

A plot of $I_0(x)$, $I_1(x)$, $K_0(x)$, and $K_1(x)$ is shown in Fig. B-2.

Another equation that is often encountered in plate and shell theory is given by

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - (ix^2 + k^2)y = 0. \quad (\text{B-5})$$

Table B-1. Values of J_0 , J_1 , Y_0 , and Y_1

(x)	$J_0(x)$	$J_1(x)$	$Y_0(x)$	$Y_1(x)$
0.0	1.0000	0.0000	$-\infty$	$-\infty$
0.5	0.9385	0.2423	-0.4445	-1.4715
1.0	0.7652	0.4401	0.0883	-0.7812
1.5	0.5118	0.5579	0.3825	-0.4123
2.0	0.2239	0.5767	0.5104	-0.1070
2.5	-0.0484	0.4971	0.4981	0.1459
3.0	-0.2601	0.3391	0.3769	0.3247
3.5	-0.3801	0.1374	0.1890	0.4102
4.0	-0.3972	-0.0660	-0.0169	0.3979
4.5	-0.3205	-0.2311	-0.1947	0.3010
5.0	-0.1776	-0.3276	-0.3085	0.1479
5.5	-0.0068	-0.3414	-0.3395	-0.0238
6.0	0.1507	-0.2767	-0.2882	-0.1750
6.5	0.2601	-0.1538	-0.1732	-0.2741
7.0	0.3001	-0.0047	-0.0260	-0.3027
7.5	0.2663	0.1353	0.1173	-0.2591
8.0	0.1717	0.2346	0.2235	-0.1581
8.5	0.0419	0.2731	0.2702	-0.0262
9.0	-0.0903	0.2453	0.2499	0.1043
9.5	-0.1939	0.1613	0.1712	0.2032
10.0	-0.2459	0.0435	0.0557	0.2490

The solution of this equation (Hetenyi 1964) for the important case of $k = 0$ is given by

$$y = C_1Z_1(x) + C_2Z_2(x) + C_3Z_3(x) + C_4Z_4(x) \tag{B-6}$$

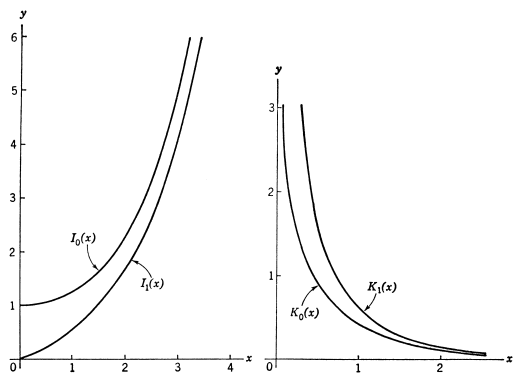


Figure B-2. A plot of modified Bessel functions. (Wylie 1960.)

where

$$Z_1(x) = \text{ber}(x) = \sum_{m=0}^{\infty} (-1)^m \frac{(x/2)^{4m}}{[(2m)!]^2}$$

$$Z_2(x) = -\text{bei}(x) = -\sum_{m=0}^{\infty} (-1)^m \frac{(x/2)^{4m+2}}{[(2m+1)!]^2}$$

$$Z_3(x) = -\frac{2}{\pi} \text{kei}(x) = \frac{Z_1(x)}{2} - \frac{2}{\pi} [R_1 + (\ln(\gamma x/2))(Z_2(x))]$$

$$Z_4(x) = -\frac{2}{\pi} \text{ker}(x) = \frac{Z_2(x)}{2} - \frac{2}{\pi} [R_2 + (\ln(\gamma x/2))(Z_1(x))]$$

$$R_1 = (x/2)^2 - \frac{h(3)}{(3!)^2} (x/2)^6 + \frac{h(5)}{(5!)^2} (x/2)^{10} - \dots$$

$$R_2 = \frac{h(2)}{(2!)^2} (x/2)^4 - \frac{h(4)}{(4!)^2} (x/2)^8 + \frac{h(6)}{(6!)^2} (x/2)^{12} - \dots$$

$$h(n) = 1 + 1/2 + 1/3 + \dots + 1/n$$

$$\gamma = 0.5772.$$

A plot of $Z_1(x)$, $Z_2(x)$, $Z_3(x)$, $Z_4(x)$, and their derivatives is shown in Fig. B-3.

B-2 Some Bessel Identities

The derivatives and integrals of Bessel functions follow a certain pattern. The identities given here are needed to solve some of the problems given in this text.

$$\frac{d}{dx} [xJ_1(x)] = xJ_0(x)$$

$$\frac{d}{dx} J_0(x) = -J_1(x)$$

$$\frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x)$$

$$\frac{d}{dx} \left[\frac{J_n(x)}{x^n} \right] = \frac{-J_{n+1}(x)}{x^n}$$

$$\frac{d^2 Z_1(x)}{dx^2} = Z_2(x) - \frac{1}{x} \frac{dZ_1(x)}{dx}$$

$$\frac{d^2 Z_2(x)}{dx^2} = -Z_1(x) - \frac{1}{x} \frac{dZ_2(x)}{dx}$$

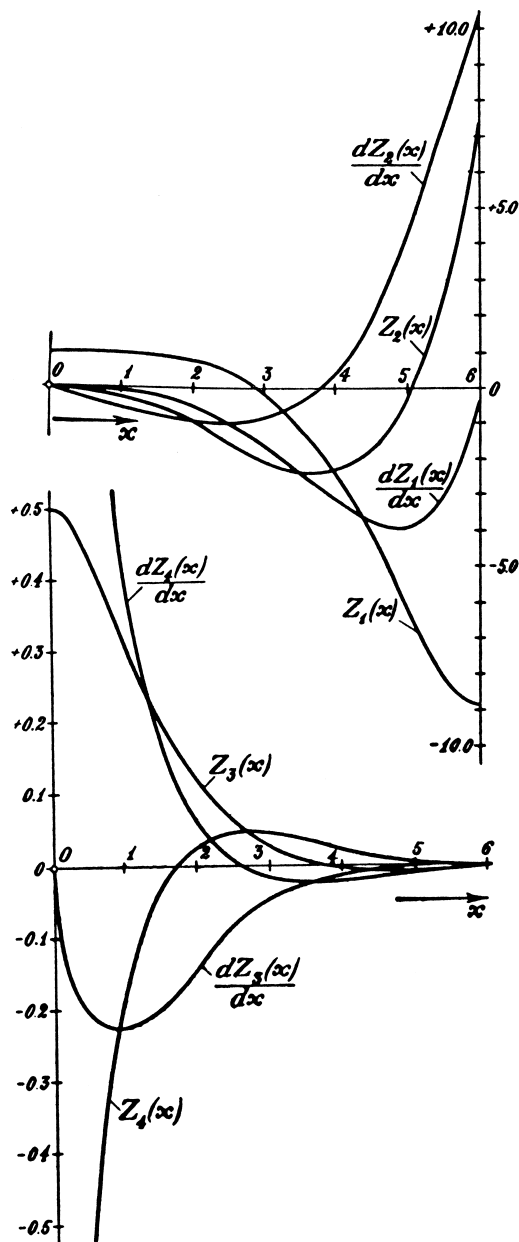


Figure B-3. Plot of functions Z . (Hetenyi 1964.)

$$\frac{d^2 Z_3(x)}{dx^2} = Z_4(x) - \frac{1}{x} \frac{dZ_3(x)}{dx}$$

$$\frac{d^2 Z_4(x)}{dx^2} = -Z_3(x) - \frac{1}{x} \frac{dZ_4(x)}{dx}$$

The last four equations are needed in the solution of circular plates on elastic foundation. In these equations the value of (kx) is needed rather than (x) in the Z functions. In this case, these equations take on the form

$$k^2 Z_1''(kx) = k^2 Z_2(x) - \frac{k}{x} Z_1'(kx)$$

$$k^2 Z_2''(kx) = -k^2 Z_1(x) - \frac{k}{x} Z_2'(kx)$$

$$k^2 Z_3''(kx) = k^2 Z_4(x) - \frac{k}{x} Z_3'(kx)$$

$$k^2 Z_4''(kx) = -k^2 Z_3(x) - \frac{k}{x} Z_4'(kx)$$

B-3 Simplified Bessel Functions

As x approaches zero, the various Bessel functions can be expressed as

$$J_k(x) = \frac{x^k}{(2^k)(k!)}$$

$$Y_k(x) = \frac{-2^k(k-1)!}{\pi} x^{-k} \quad k \neq 0$$

$$Y_0(x) = \frac{2}{\pi} \ln x$$

$$I_k(x) = \frac{x^k}{2^k k!}$$

$$K_k(x) = (2^{k-1})[(k-1)!]x^{-k} \quad k \neq 0$$

$$K_0(x) = -\ln x$$

$$Z_1(x) = 1.0 \quad Z_2(x) = -x^2/4$$

$$Z_3(x) = 1/2 \quad Z_4(x) = \frac{2}{\pi} \ln \frac{\gamma x}{2}$$

$$\frac{dZ_1(x)}{dx} = -x^3/16 \quad \frac{dZ_3(x)}{dx} = \frac{x}{\pi} \ln \frac{\gamma x}{2}$$

$$\frac{dZ_2(x)}{dx} = -x/2 \quad \frac{dZ_4(x)}{dx} = \frac{2}{\pi x}$$

As x approaches infinity, the various Bessel functions can be expressed as

$$J_k(x) = \sqrt{\frac{2}{\pi x}} \cos(x - \xi_k) \quad \xi_k = (2k + 1) \frac{\pi}{4}$$

$$Y_k(x) = \sqrt{\frac{2}{\pi x}} \sin(x - \xi_k)$$

$$I_k(x) = \frac{e^x}{\sqrt{2\pi x}}$$

$$K_k(x) = \frac{e^{-x}}{\sqrt{2x/\pi}}$$

$$Z_1(x) = \eta \cos \sigma \quad Z_2(x) = -\eta \sin \sigma$$

$$Z_3(x) = \beta \sin \tau \quad Z_4(x) = -\beta \cos \tau$$

$$\eta = \frac{1}{\sqrt{2\pi x}} e^{x/\sqrt{2}} \quad \beta = \sqrt{\frac{2}{\pi x}} e^{-x/\sqrt{2}}$$

$$\sigma = \frac{x}{\sqrt{2}} - \pi/8 \quad \tau = \frac{x}{\sqrt{2}} + \pi/8$$

$$\frac{dZ_1(x)}{dx} = \frac{\eta}{\sqrt{2}} (\cos \sigma - \sin \sigma)$$

$$\frac{dZ_2(x)}{dx} = \frac{-\eta}{\sqrt{2}} (\cos \sigma + \sin \sigma)$$

$$\frac{dZ_3(x)}{dx} = \frac{\beta}{\sqrt{2}} (\cos \tau - \sin \tau)$$

$$\frac{dZ_4(x)}{dx} = \frac{-\beta}{\sqrt{2}} (\cos \tau + \sin \tau)$$

APPENDIX C

Solution of Linear Equations

C-1 Gauss Elimination

In this method the simultaneous equations are reduced to an upper triangular matrix as shown in the following example.

	<u>operation</u>
$4x_1 + 2x_2 + 0x_3 = 4$	(1)
$2x_1 + 8x_2 + 2x_3 = 12$	(2)
<u>$0x_1 + 2x_2 + 4x_3 = 8$</u>	(3)
$4x_1 + 2x_2 + 0x_3 = 4$	(4) = (1)
$0 + 14x_2 + 4x_3 = 20$	(5) = 2(2) - (4)
<u>$0 + 0 + 24x_3 = 36$</u>	(6) = 7(3) - (5)

by back substitution,

$$x_3 = 1.5 \quad \text{from equation (6)}$$

$$x_2 = 1.0 \quad \text{from equation (5)}$$

$$x_1 = 0.5 \quad \text{from equation (4)}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.0 \\ 1.5 \end{bmatrix}$$

C-2 Gauss-Jordan Method

This method consists of reducing the simultaneous equations to a unit matrix as shown in the following example.

	<u>operation</u>
4 2 0 4	(1)
2 8 2 12	(2)
<u>0 2 4 8</u>	(3)

1.0 0.5 0.0 1	(4) = (1)/4
0.0 14.0 4.0 20	(5) = 2(2) - (1)
<u>0.0 0.5 1.0 2.0</u>	(6) = (3)/4

1.0 0.0 - 1.0 - 1.0	(7) = (4) - (6)
0.0 1.0 2/7 10/7	(8) = (5)/14
<u>0.0 0.0 6/7 9/7</u>	(9) = (6) - (5)/28

1.0 0.0 0.0 1/2	(10) = (7) + 7/6(9)
0.0 1.0 0.0 1	(11) = (8) - (9)/3
<u>0.0 0.0 1.0 3/2</u>	(12) = 7/6(9)

and
 $x_1 = 0.5$

$x_2 = 1.0$

$x_3 = 1.5$

A computer program for inverting and multiplying matrices using this method is given at the end of this appendix.

C-3 The Decomposition (Choleski) Method

If the matrix [A] in the equation

$$[A][X] = [C] \tag{C-1}$$

is a known symmetric nxn matrix and [C] is a known function, then matrix [X] can be obtained by first decomposing matrix [A] into a lower and upper matrices.

$$[A] = [L][U] \tag{C-2}$$

where,

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \dots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \dots & A_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & A_{n3} & \dots & A_{nn} \end{bmatrix}$$

$$[L] = \begin{bmatrix} L_{11} & 0 & 0 & \dots & 0 \\ L_{21} & L_{22} & L_{23} & \dots & 0 \\ L_{31} & L_{32} & L_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ L_{n1} & L_{n2} & L_{n3} & \dots & L_{nn} \end{bmatrix}$$

and

$$[U] = \begin{bmatrix} 1 & U_{12} & U_{13} & \dots & U_{1n} \\ 0 & 1 & U_{23} & \dots & U_{2n} \\ 0 & 0 & 0 & \dots & U_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

The values of $[L]$ and $[U]$ are calculated as follows

From row 1

$$A_{11} = (L_{11})(1) \quad \text{or} \quad L_{11} = A_{11}$$

$$A_{12} = (L_{11})(U_{12}) \quad \text{or} \quad U_{12} = A_{12}/L_{11}$$

$$A_{13} = (L_{11})(U_{13}) \quad \text{or} \quad U_{13} = A_{13}/L_{11}$$

$$A_{1n} = (L_{11})(U_{1n}) \quad \text{or} \quad U_{1n} = A_{1n}/L_{11}$$

From row 2

$$A_{21} = (L_{21})(1) \quad \text{or} \quad L_{21} = A_{21}$$

$$A_{22} = (L_{21})(U_{12}) + (L_{22})(1) \quad \text{or} \quad L_{22} = A_{22} - (L_{21})(U_{12})$$

$$A_{23} = (L_{21})(U_{13}) + (L_{22})(U_{23}) \quad \text{or} \quad U_{23} = [A_{23} - (L_{21})(U_{13})]/(L_{22})$$

$$A_{2n} = (L_{21})(U_{1n}) + (L_{22})(U_{2n}) \quad \text{or} \quad U_{2n} = [A_{2n} - (L_{21})(U_{1n})]/(L_{22})$$

The values in the remaining rows are obtained by the same procedure. The above calculations can be written in a more compact form as

$$L_{ij} = A_{ij} - \sum_{k=1}^{j-1} L_{ik} U_{kj} \quad \text{where } i \geq j$$

$$U_{ij} = \frac{A_{ij} - \sum_{k=1}^{j-1} L_{ik} U_{kj}}{L_{ii}} \quad \text{where } i < j$$

$$U_{ii} = 1$$

Based on the above, Eq. (C-1) can be written as

$$[L][U][X] = [C] \quad (C-3)$$

Define

$$[U][X] = [Z] \quad (C-4)$$

Then Eq. (C-3) can be written as

$$[L][Z] = [C] \quad (C-5)$$

Solution of Eq. (C-1) is accomplished by first solving for $[Z]$ in Eq. (C-5) by forward substitution. The unknown vector $[X]$ in Eq. (C-1) can then solved by substituting $[Z]$ into Eq. (C-4) and solving for $[X]$ by back substitution. The substitution. This is illustrated in the following example

Let

$$[A] = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 4 \end{bmatrix} \quad \text{and} \quad [C] = \begin{bmatrix} 4,000 \\ 12,000 \\ 8,000 \end{bmatrix}$$

From Eq. (C-2),

$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} 1 & U_{12} & U_{13} \\ 0 & 1 & U_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

The values of L and U are obtained as follows

Row 1

$$A_{11} = (L_{11})(1) \quad \text{or} \quad L_{11} = 4,000$$

$$A_{12} = (L_{11})(U_{12}) \quad \text{or} \quad U_{12} = 2,000/4,000 = 0.500$$

$$A_{13} = (L_{11})(U_{13}) \quad \text{or} \quad U_{13} = 0.000/4,000 = 0.000$$

Row 2

$$A_{21} = (L_{21})(1) \quad \text{or} \quad L_{21} = 2,000$$

$$A_{22} = (L_{21})(U_{12}) + (L_{22})(1) \quad \text{or} \quad L_{22} = A_{22} - (L_{21})(U_{12})$$

$$L_{22} = 8.000 - (2.000)(0.500) = 7.000$$

$$A_{23} = (L_{21})(U_{13}) + (L_{22})(U_{23}) \quad \text{or} \quad U_{23} = [A_{23} - (L_{21})(U_{13})]/(L_{22})$$

$$U_{23} = [2.000 - (2.000)(0.000)]/7.000 = 0.286$$

Row 3

$$A_{31} = (L_{31})(1) \quad \text{or} \quad L_{31} = 0.000$$

$$A_{32} = (L_{31})(U_{12}) + (L_{32})(1) \quad \text{or} \quad L_{32} = A_{32} - (L_{31})(U_{12})$$

$$L_{32} = 2.000 - (0.000)(0.500) = 2.000$$

$$A_{33} = (L_{31})(U_{13}) + (L_{32})(U_{23}) + (L_{33})(1) \quad \text{or} \quad L_{33} = A_{33} - (L_{31})(U_{13}) - (L_{32})(U_{23})$$

$$L_{33} = 4.000 - (0.000)(0.000) - (2.000)(0.286) = 3.428$$

From Eq. (C-2),

$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4.000 & 0 & 0 \\ 2.000 & 7.000 & 0 \\ 0.000 & 2.000 & 3.428 \end{bmatrix} \begin{bmatrix} 1 & 0.500 & 0.000 \\ 0 & 1 & 0.286 \\ 0 & 0 & 1 \end{bmatrix}$$

And Eq. (C-5) can be written as

$$\begin{bmatrix} 4.000 & 0 & 0 \\ 2.000 & 7.000 & 0 \\ 0.000 & 2.000 & 3.428 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 4.000 \\ 12.000 \\ 8.000 \end{bmatrix}$$

from the first row, $Z_1 = 1.000$

from the second row, $Z_2 = 1.429$

from the third row, $Z_3 = 1.500$

Equation (C-4) then becomes

$$\begin{bmatrix} 1 & 0.500 & 0.000 \\ 0 & 1 & 0.286 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1.000 \\ 1.429 \\ 1.500 \end{bmatrix}$$

from the third row, $X_3 = 1.500$

from the second row, $X_2 = 1.000$

from the first row, $X_1 = 0.500$

Hence, the value of the unknown quantity [X] is

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.0 \\ 1.5 \end{bmatrix}$$

This method, while cumbersome to use manually, is very efficient when using computer programs. The mathematical computer formulation is available in the literature.

Program for Matrix Inversion and Multiplication

```

10 REM
20 REM – THE NAME OF THIS PROGRAM IS “MATRIX”
30 REM
40 REM – MATRIX INVERSION IS BY MODIFIED GAUSS-JORDAN
  METHOD
50 REM
60 REM – DATE OF PROGRAM “FEBRUARY, 2003”. BY MAAN JAWAD
70 REM
80 REM
90 REM
100 PRINT “THIS PROGRAM CAN BE USED TO”
110 PRINT
120 PRINT “1. INVERT A MATRIX”
130 PRINT
140 PRINT “2. INVERT A MATRIX THEN MULTIPLY IT BY ANOTHER
  MATRIX”
150 PRINT
160 PRINT “ 3. MULTIPLY TWO MATRICES TOGETHER”
170 PRINT
180 DEFDBL A–Z : DEFINT I – L, C, R
190 PRINT “DO YOU WANT TO INVERT A MATRIX?: Y OR N”
200 INPUT Z$
210 IF Z$ = “N” THEN 1450
220 PRINT “MATRIX INVERSION”
230 PRINT
240 REM – MATRIX IS SQUARE
250 PRINT “DIMENSION OF SQUARE MATRIX TO BE INVERTED (R,R)”;
260 INPUT R,R
270 REM – ARRAYS A() &B() SET TO THE DIMENSIONS OF THE
  MATRIX

```

```
280 DIM A(R,R), B(R,R)
290 PRINT
300 PRINT "MATRIX ELEMENTS:"
310 REM - ENTER MATRIX ELEMENTS
320 FOR J = 1 TO R
330 PRINT
340 PRINT "ROW"; J
350 FOR I = 1 TO R
360 PRINT "VALUE COLUMN"; I;
370 INPUT A(J,I)
380 NEXT I
390 B(J,J) = 1
400 NEXT J
410 PRINT
420 PRINT "ORIGINAL MATRIX IS INPUT AS"
430 PRINT
440 R1 = R
450 C1 = R
460 FOR I = 1 TO R
470 FOR J = 1 TO R
480 PRINT A(I,J); " ";
490 NEXT J
500 PRINT
510 NEXT I
520 PRINT : INPUT "DO YOU WANT TO REVISE ANY OF THE ABOVE
    DATA, Y/N"; Y$ : PRINT
530 IF Y$ = "Y" THEN 540 ELSE 580
540 INPUT "ROW NO."; J
550 INPUT "COLUMN NO."; I
560 INPUT "REVISED VALUE"; A(J,I)
570 PRINT : GOTO 430
580 PRINT "DO YOU WANT TO PRINT ORIGINAL MATRIX?: Y OR N"
590 INPUT Q$
600 IF Q$ = "N" THEN 730
610 LPRINT
620 LPRINT "ORIGINAL MATRIX IS INPUT AS"
630 LPRINT
640 R1 = R
650 C1 = R
660 FOR I = 1 TO R
670 FOR J = 1 TO R
680 LPRINT A(I,J); " ";
```

```
690 NEXT J
700 LPRINT
710 NEXT I
720 REM
730 REM – STATEMENTS 660 TO 950 INVERT MATRIX
740 REM
750 FOR J = 1 TO R
760 I = J – 1
770 I = I + 1
780 IF I > R THEN 810
790 IF A(I,J) <> 0 THEN 830
800 GOTO 770
810 PRINT “SINGULAR MATRIX”
820 GOTO 1430
830 FOR K = 1 TO R
840 S = A(J,K)
850 A(J,K) = A(I,K)
860 A(I,K) = S
870 S = B(J,K)
880 B(J,K) = B(I,K)
890 B(I,K) = S
900 NEXT K
910 T = 1/A(J,J)
920 FOR K = 1 TO R
930 A(J,K) = T*A(J,K)
940 B(J,K) = T*B(J,K)
950 NEXT K
960 FOR L = 1 TO R
970 IF L = J THEN 1030
980 T = –A(L,J)
990 FOR K = 1 TO R
1000 A(L,K) = A(L,K) + T*A(J,K)
1010 B(L,K) = B(L,K) + T*B(J,K)
1020 NEXT K
1030 NEXT L
1040 NEXT J
1050 PRINT
1060 PRINT “INVERSE OF MATRIX IS”
1070 PRINT
1080 REM
1090 REM – PRINT RESULTANT MATRIX
1100 REM
```

```
1110 FOR I = 1 TO R
1120 FOR J = 1 TO R
1130 PRINT B(I,J); " ";
1140 NEXT J
1150 REM - ADVANCE OUTPUT DEVICE TO PRINT NEXT LINE
1160 PRINT
1170 NEXT I
1180 PRINT "DO YOU WANT TO PRINT INVERSTED MATRIX ?; Y OR N"
1190 INPUT Q$
1200 IF Q$ = "N" THEN 1330
1210 LPRINT
1220 LPRINT "INVERSE OF MATRIX IS"
1230 LPRINT
1240 FOR I = 1 TO R
1250 FOR J = 1 TO R
1260 LPRINT B(I,J); " ";
1270 NEXT J
1280 LPRINT
1290 NEXT I
1300 REM
1310 REM - THE FOLLOWING IS FOR MULTIPLYING INVERTED
      MATRIX
1320 REM
1330 PRINT
1340 PRINT "DO YOU WANT TO MULTIPLY INVERTED MATRIX?:
      Y OR N"
1350 INPUT Z$
1360 IF Z$ = "Y" THEN 1920
1370 PRINT
1380 PRINT "DO YOU WANT TO INVERT ANOTHER MATRIX?: Y OR N"
1390 INPUT Y$
1400 IF Y$ = "Y" THEN 100
1410 GOTO 2700
1420 REM
1430 REM - THE FOLLOWING PART IS FOR MATRIX MULTIPLICATION
1440 REM
1450 PRINT "MATRIX MULTIPLICATION"
1460 PRINT
1470 PRINT "DIMENSION OF MATRIX 1 ( R,C)";
1480 INPUT R1,C1
1490 REM - ARRAYS B() AND D() SET TO DIMENSIONS OF MATRICES
1500 DIM B(R1,C1)
```

```
1510 PRINT
1520 REM
1530 REM-ENTER MATRIX VALUES
1540 REM
1550 PRINT "MATRIX 1:"
1560 FOR I = 1 TO R1
1570 PRINT
1580 PRINT "ROW"; I
1590 FOR J = 1 TO C1
1600 PRINT "VALUE COLUMN"; J;
1610 INPUT B(I,J)
1620 NEXT J
1630 NEXT I
1640 PRINT
1650 PRINT "MATRIX 1 IS INPUT AS"
1660 PRINT
1670 FOR I = 1 TO R1
1680 FOR J = 1 TO C1
1690 PRINT B(I,J); " ";
1700 NEXT J
1710 PRINT
1720 NEXT I
1730 PRINT
1740 PRINT : INPUT "DO YOU WANT TO REVISE ANY OF THE ABOVE
      DATA, Y/N"; X$: PRINT
1750 IF X$ = "Y" THEN 1760 ELSE 1800
1760 INPUT "ROW NO."; I
1770 INPUT "COLUMN NO."; J
1780 INPUT "REVISED VALUE"; B(I,J)
1790 PRINT : GOTO 1660
1800 PRINT "DO YOU WANT TO PRINT MATRIX 1?: Y OR N"
1810 INPUT Q$
1820 IF Q$ = "N" THEN 1920
1830 LPRINT "MATRIX 1 IS INPUT AS"
1840 LPRINT
1850 FOR I = 1 TO R1
1860 FOR J = 1 TO C1
1870 LPRINT B(I,J); " ";
1880 NEXT J
1890 LPRINT
1900 NEXT I
1910 PRINT
```

```
1920 PRINT "MATRIX 2:"
1930 PRINT "DIMENSION OF MATRIX 2 (R,C)";
1940 INPUT R2,C2
1950 REM
1960 REM - # OF COLUMNS IN MATRIX 1 MUST EQUAL # OF ROWS IN
      MATRIX 2
1970 REM
1980 IF C1 = R2 THEN 2010
1990 PRINT CHR$(7) "CANNOT BE MULTIPLIED; OTHER DIMENSIONS
      NECESSARY"
2000 GOTO 1920
2010 DIM D(R2,C2)
2020 FOR I = 1 TO R2
2030 PRINT
2040 PRINT "ROW"; I
2050 FOR J = 1 TO C2
2060 PRINT "VALUE COLUMN"; J;
2070 INPUT D(I,J)
2080 NEXT J
2090 NEXT I
2100 PRINT
2110 PRINT "MATRIX 2 IS INPUT AS"
2120 PRINT
2130 FOR I = 1 TO R2
2140 FOR J = 1 TO C2
2150 PRINT D(I,J); " ";
2160 NEXT J
2170 PRINT
2180 NEXT I
2190 PRINT
2200 PRINT : INPUT "DO YOU WANT TO REVISE ANY OF THE ABOVE
      DATA, Y/N"; W$: PRINT
2210 IF W$ = "Y" THEN 2220 ELSE 2260
2220 INPUT "ROW NO."; I
2230 INPUT "COLUMN NO."; J
2240 INPUT "REVISED VALUE"; D(I,J)
2250 PRINT : GOTO 2120
2260 PRINT "DO YOU WANT TO PRINT MATRIX 2?: Y OR N"
2270 INPUT Q$
2280 IF Q$ = "N" THEN 2390
2290 LPRINT "MATRIX 2 IS INPUT AS"
2300 FOR I = 1 TO R2
```

```
2310 FOR J = 1 TO C2
2320 LPRINT D(I,J); " ";
2330 NEXT J
2340 LPRINT
2350 NEXT I
2360 REM
2370 REM – PERFORM MATRIX MULTIPLICATION, PRINT RESULTANT
      MATRIX
2380 REM
2390 PRINT "THE RESULT OF MULTIPLYING MATRIX 1 * MATRIX 2 IS"
2400 PRINT
2410 FOR I = 1 TO R1
2420 FOR J = 1 TO C2
2430 S = 0
2440 FOR K = 1 TO C1
2450 S = S + B (I,K)*D(K,J)
2460 NEXT K
2470 PRINT S; " ";
2480 NEXT J
2490 REM
2500 REM – ADVANCE OUTPUT TO PRINT NEXT ROW
2510 REM
2520 PRINT
2530 NEXT I
2540 PRINT "DO YOU WANT TO PRINT MATRIX 1 * MATRIX 2?: Y OR N"
2550 INPUT Q$
2560 IF Q$ = "N" THEN 2690
2570 LPRINT "THE RESULT OF MULTIPLYING MATRIX 1 * MATRIX 2
      IS"
2580 LPRINT
2590 FOR I = 1 TO R1
2600 FOR J = 1 TO C2
2610 S = 0
2620 FOR K = 1 TO C1
2630 S = S + B (I,K)*D(K,J)
2640 NEXT K
2650 LPRINT S; " ";
2660 NEXT J
2670 LPRINT
2680 NEXT I
2690 PRINT : PRINT "***** END OF PROGRAM *****"
2700 END
```

APPENDIX D

Conversion Factors

Pressure Units					
	1 psi	1 N/mm ²	1 bar	1 KPa	1 kgf/cm ²
psi	1.0000	145.0	14.50	0.1450	14.22
N/mm ²	0.006895	1.000	0.1000	0.0010	0.09807
bars	0.06895	10.000	1.000	0.0100	0.9807
KPa	6.895	1000.0	100.00	1.000	98.07
kgf/cm ²	0.0703	10.20	1.020	0.0102	1.000

1 N/mm² = 1 MPa.

Modulus of Elasticity Units				
	1 ksi	1 KN/mm ²	1 MPa	1 kgf/mm ²
ksi	1.000	145.0	0.1450	1.422
KN/mm ²	0.006895	1.000	0.001	0.009807
MPa	6.895	1000.00	1.000	9.807
kgf/mm ²	0.7033	102.0	0.1020	1.000

Force Units			
	1 lb	1 kgf	1 N
lb	1.000	2.205	0.2248
kgf	0.454	1.000	0.1020
N	4.448	9.807	1.0000

Answers to Selected Problems

1-2 $\sigma_1 = 11,945 \text{ psi}$, $\sigma_2 = -1360 \text{ psi}$, $\tau_{12} = 6650 \text{ psi}$

1-4 $\sigma_x = 92.2 \text{ MPa}$, $\sigma_y = 124.9 \text{ MPa}$

1-15 $t = 2.74 \text{ inch}$

2-5

$$M_x = p \sum_{m=1,3}^{\infty} \left[\frac{m^2 \pi^2}{a^2} H_m (1 - \mu) \cosh \frac{m \pi y}{a} \right. \\ \left. + \frac{m^2 \pi^2}{a^2} G_m H_m \nu (1 - \mu) \sinh \frac{m \pi y}{a} \right. \\ \left. - \frac{2 \mu m \pi}{a} G_m H_m \cosh \frac{m \pi y}{a} + \frac{4 a^2}{m^3 \pi^3} \right] \sin \frac{m \pi x}{a}$$

$$M_y = p \sum_{m=1,3}^{\infty} \left[\frac{m^2 \pi^2}{a^2} H_m (\mu - 1) \cosh \frac{m \pi y}{a} \right. \\ \left. + \frac{m^2 \pi^2}{a^2} G_m H_m \nu (\mu - 1) \sinh \frac{m \pi y}{a} \right. \\ \left. - \frac{2 m \pi}{a} G_m H_m \cosh \frac{m \pi y}{a} + \frac{4 \mu a^2}{m^3 \pi^3} \right] \sin \frac{m \pi x}{a}$$

$$H_m = \frac{-4 a^4}{m^5 \pi^5 \left(\cosh \frac{m \pi b}{2 a} + G_m \frac{b}{2} \sinh \frac{m \pi b}{2 a} \right)}$$

$$G_m = \frac{\frac{m \pi}{a} \sinh \frac{m \pi b}{2 a}}{\sinh \frac{m \pi b}{2 a} + \frac{m \pi}{a} \cosh \frac{m \pi b}{2 a}}$$

2-6

$$w = \sum_{m=1}^{\infty} \left(B_m \cosh \frac{m\pi y}{a} + C_m y \sinh \frac{m\pi y}{a} \right) \sin \frac{m\pi x}{a}$$

where

$$B_m = E_m / C_1, \quad C_m = -\frac{2}{b} B_m \coth \frac{m\pi b}{2a}$$

$$C_1 = \left(\frac{m^2 \pi^2}{a^2} - \frac{2m\pi}{a} C_2 \right) \cosh \frac{m\pi b}{2a} - \frac{m^2 \pi^2}{a^2} C_2 \frac{b}{2} \sinh \frac{m\pi b}{2a}$$

$$C_2 = \frac{2}{b} \coth \frac{m\pi b}{2a}$$

2-12 $M = 7,620$ inch-lb/inch3-3 $= 8670$ psi, Max $w = 0.23$ inch.3-4 $= 12,330$ psi

3-6

$$M_r = \frac{p}{16} (3 + \mu) (a^2 - r^2) - \frac{pb^2}{4} (K_1 + K_2 - K_3 - K_4 - 1)$$

where

$$K_1 = (1 + \mu) \left(\frac{3 + \mu}{2(1 + \mu)} - \frac{b^2}{a^2 - b^2} \ln b/a - 1/2 \right)$$

$$K_2 = \frac{1 - \mu}{r} \left(\frac{1 + \mu}{1 - \mu} \frac{a^2 b^2}{a^2 - b^2} \ln b/a \right)$$

$$K_3 = (1 + \mu) \ln r/a$$

$$K_4 = \frac{(3 + \mu)}{4} \left(\frac{a^2 - r^2}{r^2} \right)$$

$$M_t = \frac{p}{16} [a^2 (3 + \mu) - r^2 (1 + 3\mu)] - \frac{pb^2}{4} (K_1 + K_2 - K_3 + K_5 - \mu)$$

$$K_5 = \frac{(3 + \mu)}{4} \left(\frac{a^2 + r^2}{r^2} \right)$$

$$5-4 \quad M_p = pL^2/8$$

$$5-6 \quad M_p = pL^2/144$$

$$5-11 \quad M_p = 157.3 \, p$$

$$8-4 \quad t = 0.90 \text{ inch}$$

$$8-5 \quad N_\phi = -p_o R/2$$

$$N_\theta = -p_o R(\cos^2 \phi - 1/2)$$

$$8-6$$

$$N_\phi = \frac{-\gamma R^2}{6} \left(\frac{3H}{R} + 1 - \frac{2 \cos^2 \phi}{1 + \cos \phi} \right)$$

$$N_\theta = \frac{-\gamma R^2}{6} \left(\frac{3H}{R} - 1 - \frac{6 - 4 \cos^2 \phi}{1 + \cos \phi} \right)$$

$$8-9$$

$$\max N_s = \frac{-847\gamma L^2 \sin \alpha}{432} \quad \text{at } s = L/12$$

$$\max N_\theta = \frac{\gamma L^2}{4} \sin \alpha \quad \text{at } s = L/2$$

$$9-4 \quad A = 5.76 \text{ in}^2$$

$$9-5$$

$$t_1 = 0.21 \text{ inch}, \quad t_2 = 0.82 \text{ inch}$$

$$t_3 = 0.98 \text{ inch}, \quad t_4 = 0.69 \text{ inch}$$

$$A = 1.22 \text{ in}^2$$

$$10-1 \quad \text{Max } M_x = 0.322 \, Q_o/B \quad \text{at } x = 0.61\sqrt{rt}$$

$$10-2 \quad \text{At section } a-a \, M_o = 0 \text{ and } H_o = 0.0195D\beta^3$$

$$10-3 \quad M_a = 14.95/\beta \text{ and } M_b = 44.97/\beta$$

$$10-6$$

$$p = 3.49 \text{ psi}$$

$$M_o = 0.1074P \text{ inch-lb/inch}, \quad Q_o = 1.855P \text{ lb/inch.}$$

10-7 Discontinuity moments and forces at junction are

moment in top cylinder = 595.4 inch-lb/inch (comp. on outside)

moment in bottom cylinder = 163.1 inch-lb/inch (comp. on outside)

moment in plate = 758.5 inch-lb/inch (tension on top)

horizontal force in top cylinder = 495.9 lbs compressive

horizontal force in bottom cylinder = 109.2 lbs tensile

horizontal force in plate = 386.7 lbs tensile

11-2 $M_o = 1886$ inch-lb/inch, $H_o = 1200$ lb/inch

13-3 $t = 5/16$ inch.

14-2 $t = 0.73$ inch.

14-3 $p = 148.9$ psi.

A-2

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi} \sum_{m=2,4}^{\infty} \frac{\cos mx}{(m-1)(m+1)} + \frac{1}{2} \sin x$$

References

- American Association of State Highway and Transportation Officials. 1996. Standard Specifications for Highway Bridges HB-16. Washington, DC: AASHTO.
- American Concrete Institute. 2002. Building Code Requirements for Structural Concrete. ACI-318. Chicago, IL: ACI.
- American Institute of Steel Construction. 1991. Manual of Steel Construction-Allowable Stress Design. Chicago, IL: AISC.
- American Iron and Steel Institute. 1981. Steel Penstocks and Tunnel Liners. Washington, DC: AISI.
- American Petroleum Institute. 2002. Design and Construction of Large, Welded, Low-Pressure Storage Tanks—API 620. Washington, DC: API.
- American Society of Civil Engineers. 1960. Design of Cylindrical Concrete Shell Roofs. Manual No. 31. Reston, VA: ASCE.
- American Society of Mechanical Engineers. 2001a. Pressure Vessel Code, Section VIII, Division 1, New York, NY: ASME.
- American Society of Mechanical Engineers. 2001b. Pressure Vessel-Alternate Rules, Section VIII, Division 2, New York, NY: ASME.
- American Society of Mechanical Engineers. 2001c. Fiber-Reinforced Plastic Pressure Vessels, Section X, New York, NY: ASME.
- Baker, E. H., Cappelli, A. P., Kovalevsky, L., Rish, F. L., Verette, R. M. 1968. Shell Analysis Manual—NASA 912. Washington, DC: National Aeronautics and Space Administration.
- Becker, H. July 1957. Handbook of Structural Stability—Part II—Buckling of Composite Elements NACA PB 128 305. Washington, DC: National Advisory Committee for Aeronautics.
- Beer, F. P., Johnston Jr., E. R., and Dewolf, J. T. 2001. Mechanics of Materials. New York, NY: McGraw Hill.
- Billington, D. P. 1982. Thin Shell Concrete Structures. New York, NY: McGraw Hill.
- Bloom, F. and Coffin, D. 2001. Handbook of Thin Plate Buckling and Postbuckling. Boca Raton, FL: Chapman and Hall/CRC.
- Boley, A. B., and Weiner, J. H. 1997. Theory of Thermal Stresses. New York, NY: Dover Publications.
- Bowman, F. 1977. Introduction to Bessel Functions. New York, NY: Dover Publications.
- Buchert, K. P. 1964. Stiffened Thin Shell Domes. AISC Engineering Journal. Chicago, IL: AISC.

- Buchert, K. P. 1966. Buckling Considerations in the Design and Construction of Doubly Curved Space Structures. International Conference on Space Structures, 1966—F8. England: University of Surry.
- Burgreen, D. 1971. Elements of Thermal Stress Analysis. Jamaica, NY: C.P. Press.
- Chattarjee, B. K. 1971. Theory and Design of Concrete Shells. New York, NY: Gordon and Breach.
- Cheung, Y. K. 1976. Finite Strip Method in Structural Analysis. New York, NY: Pergamon Press.
- Dwight, H. B. 1972. Tables of Integrals and Other Mathematical Data. New York, NY: Macmillan.
- Faupel, J. H. and Fisher, F. E. 1981. Engineering Design. New York, NY: Wiley-Interscience.
- Flügge, W. 1967. Stresses in Shells. New York, NY: Springer-Verlag.
- Gallagher, R. H. 1975. Finite Element Analysis. Englewood Cliffs, NJ: Prentice Hall.
- Gerard, G. August 1957. Handbook of Structural Stability—Part IV—Failure of Plates and Composite Elements. NACA N62-55784. Washington, DC: National Advisory Committee for Aeronautics.
- Gerard, G. August 1957. Handbook of Structural Stability—Part V—Compressive Strength of Flat Stiffened Panels. NACA PB 185 629. Washington, DC: National Advisory Committee for Aeronautics.
- Gerard, G. 1962. Introduction to Structural Stability Theory. New York, NY: McGraw Hill.
- Gerard, G. and Becker, H. July 1957. Handbook of Structural Stability—Part I—Buckling of Flat Elements. NACA PB 185 628. Washington, DC: National Advisory Committee for Aeronautics.
- Gerard, G. and Becker, H. 1957. Handbook of Structural Stability—Part III—Buckling of Curved Plates and Shells. NACA TN 3783. Washington, DC: National Advisory Committee for Aeronautics.
- Gibson, J. E. 1965. Linear Elastic Theory of Thin Shells. New York, NY: Pergamon Press.
- Gibson, J. E. 1968. The Design of Shell Roofs. London: E & F. N. Spon Ltd.
- Gorman, D. J. 1982. Free Vibration Analysis of Rectangular Plates. New York, NY: Elsevier.
- Gould, P. L. 1988. Analysis of Plates and Shells. New York, NY: Springer-Verlag.
- Grandin, H. Jr. 1986. Fundamentals of the Finite Element Method. New York, NY: Macmillan.
- Harrenstien, H. P. and Alsmeyer, W. C. 1959. Structural Behavior of a Plate Resembling a Constant Thickness Bridge Abutment Wingwall. Engineering Experiment Station Bulletin 182. Ames, Iowa. Iowa State University.
- Hartog, J. P. Den, 1984. Mechanical Vibrations. New York, NY: Dover Publications.
- Heins, C. P. 1976. Applied Plate Theory for the Engineer. Lexington, MA: Lexington Books.
- Hetenyi, M. 1964. Beams on Elastic Foundation. Ann Arbor, MI: University of Michigan Press.
- Hildebrand, F. 1964. Advanced Calculus for Applications. Englewood Cliffs, NJ: Prentice Hall.
- Iyengar, N. G. R. 1988. Structural Stability of Columns and Plates. New York, NY: John Wiley and Sons.
- Jawad, M. H. 1980. Design of Conical Shells Under External Pressure. Journal of Pressure Vessel Technology, Volume 102. New York, NY: ASME.
- Jawad, M. H. and Farr, J. R. 1989. Structural Analysis and Design of Process Equipment. New York, NY: John Wiley and Sons.
- Jones, L. L. 1966. Ultimate Load Analysis of Reinforced and Prestressed Concrete Structures. New York, NY: F. Unger Publishing.
- Jones, R. M. 1975. Mechanics of Composite Materials. New York, NY: Harper and Row.
- Kalnins, A. 1964. Effect of Bending on Vibration of Spherical Shells. Pages 74–81. Journal of Acoustic Society of America.

- Kelkar, V. S. and Sewell, R. T. 1987. *Fundamentals of the Analysis and Design of Shell Structures*. Englewood Cliffs, NJ: Prentice Hall.
- Kellogg, The M. W. Company. 1961. *Design of Piping Systems*. New York, NY: John Wiley and Sons.
- Koiter, W. T. 1943. The Effective Width of Flat Plates for Various Longitudinal Edge Conditions at Loads Far Beyond the Buckling Load. Rep Number 5287. The Netherlands. National Luchtvaart Laboratorium.
- Kollar, L. and Dulacska, E. 1984. *Buckling of Shells for Engineers*. New York, NY: John Wiley and Sons.
- Kraus, H. 1967. *Thin Elastic Shells*. New York, NY: John Wiley and Sons.
- Leissa, A. W. 1969. *Vibration of Plates*. NASA Publication SP-160. Washington, DC: NASA.
- Leissa, A. W. 1973. *Vibration of Shells*. NASA Publication SP-288. Washington, DC: NASA.
- Love, A. E. H. 1944. *A Treatise on the Mathematical Theory of Elasticity*. New York, NY: Dover Publications.
- Marguerre, K. 1937. The Apparent Width of the Plate in Compression. Technical Memo Number 833. Washington, DC. National Advisory Committee for Aeronautics.
- Markus, S. 1988. *The Mechanics of Vibrations of Cylindrical Shells*. New York, NY: Elsevier.
- McFarland, D. E., Smith, B. L., and Bernhart, W. D. 1972. *Analysis of Plates*. New York, NY: Macmillan.
- Moody, W. T. 1970. Moments and Reactions for Rectangular Plates — Engineering Monograph No. 27. Washington, DC: U.S. Department of the Interior.
- Moy, S. S. J. 1981. *Plastic Methods for Steel and Concrete Structures*. New York, NY: John Wiley and Sons.
- Niordson, F. I. N. 1947. Buckling of Conical Shells Subjected to Uniform External Lateral Pressure. Transactions of the Royal Institute of Technology, No. 10. Stockholm: Royal Institute of Technology.
- O'Donnell, W. J., and Langer, B. F. 1962. Design of Perforated Plates. Journal of Engineering for Industry. New York, NY: ASME.
- Perry, C. L. 1950. The Bending of Thin Elliptic Plates. Proceedings of Symposia in Applied Mathematics, Volume III, P.131. New York, NY: McGraw Hill.
- Pilkey, W. and Pin Yu Chang. 1978. *Modern Formulas for Statics and Dynamics*. New York, NY: McGraw Hill.
- Potter, M. C. 1978. *Mathematical Methods in the Physical Sciences*. Englewood Cliff, NJ: Prentice Hall.
- Raetz, R. V. 1957. An Experimental Investigation of the Strength of Small-Scale Conical Reducer Sections Between Cylindrical Shells Under External Hydrostatic Pressure. U.S. Department of the Navy, David Taylor Model Basin. Report 1187. Washington, DC: U.S. Navy.
- Rocky, K. C., Evans, H. R., Griffiths, D. W., Nethercot, D. A. 1975. *The Finite Element Method*. New York, NY: John Wiley and Sons.
- Segerlind, L. J. 1976. *Applied Finite Element Analysis*. New York, NY: John Wiley and Sons.
- Seide, P. 1962. A Survey of Buckling Theory and Experiment for Circular Conical Shells of Constant Thickness. NASA Publication IND-1510. Ohio: NASA.
- Seide, P. 1975. *Small Deformations in Thin Shells*. Leyden: Noordhoff Int. Publishing.
- Seide, P. 1981. Stability of Cylindrical Reinforced Concrete Shells. In *Concrete Shell Buckling*. ACI SP-67. Chicago, IL: American Concrete Institute.
- Shenk, A. 1997. *Calculus and Analytic Geometry*. Palo Alto, California: Addison-Wesley Publishing.

- Sherbourne, A. N. 1961. Elastic Postbuckling Behavior of a Simply Supported Circular Plate. *Journal of Mech. Eng. Sci.*, Vol. 3, 133–141.
- Sokolnikoff, I. S. 1956. *Mathematical Theory of Elasticity*. New York, NY: McGraw Hill.
- Sturm, R. G. A Study of the Collapsing Pressure of Thin-Walled Cylinders. *Engineering Experiment Station Bulletin 329*. Urbana, IL: The University of Illinois.
- Swanson, H. S., Chapton, H. J., Wilkinson, W. J., King, C. L., and Nelson, E. D. June 1955. Design of Wye Branches for Steel Pipe. New York, NY: *Journal of AWWA*.
- Szilar, R. 1974. *Theory and Analysis of Plates-Classical and Numerical Methods*. Englewood Cliffs, NJ: Prentice Hall.
- Timoshenko, S. P. 1983. *History of Strength of Materials*. New York, NY: Dover Publications.
- Timoshenko, S. P. and Gere, J. M. 1961. *Theory of Elastic Stability*. New York, NY: McGraw Hill.
- Timoshenko, S. P. and Goodier, J. N. 1951. *Theory of Elasticity*. New York, NY: McGraw Hill.
- Timoshenko, S. P. and Woinowsky-Krieger, S. 1959. *Theory of Plates and Shells*. New York, NY: McGraw Hill.
- Troitsky, M. S. 1987. *Orthotropic Bridges-Theory and Design*. Cleveland, OH: The James Lincoln Arc Welding Foundation.
- Tubular Exchanger Manufacturers Association. 1988. *Standards of the Tubular Exchanger Manufacturers Association*. Terrytown, NY: TEMA.
- Ugural, A. C. 1998. *Stresses in Plates and Shells*, New York, NY: McGraw Hill.
- Ventsel, E. and Krauthammer, T. 2001. *Thin Plates and Shells*. New York, NY: Marcel Dekker.
- Von Karman, Th. and Tsien, Hsue-Shen. 1939. The Buckling of Spherical Shells by External Pressure. In *Pressure Vessel and Piping Design-Collected Papers 1927–1959*. New York, NY: ASME.
- Wang, C. K. 1986. *Structural Analysis on Microcomputers*. New York, NY: Macmillan.
- Weaver, W. Jr. and Johnston, P. R. 1984. *Finite Elements for Structural Analysis*. Englewood Cliffs, NJ: Prentice Hall.
- Winter, G., Urquhart, L. C., O'Rourke, C. E., and Nelson, A. H. 1964. *Design of Concrete Structures*, New York, NY: McGraw Hill.
- Wood, R. H. 1961. *Plastic and Elastic Design of Slabs and Plates*. New York, NY: The Ronald Press Company.
- Wylie, C. R. Jr. 1972. *Advanced Engineering Mathematics*. New York, NY: John Wiley and Sons.
- Young, W. C. and Budynas, R. G. 2001. *Roark's Formulas for Stress and Strain*. New York, NY: McGraw Hill.
- Zick, L. P. and St. Germain, A. R. May 1963. Circumferential Stresses in Pressure Vessel Shells of Revolution. *Journal of Engineering for Industry*. New York, NY: ASME.
- Zienkiewicz, O. C. 1977. *The Finite Element Method*. New York, NY: McGraw Hill.

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